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THE FOURIER METHOD AND THE PROBLEM OF ITS COMPUTER IMPLEMENTATION

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The computer implementation of the Fourier method for solution of linear boundary value problems for PDE of the mathematical physics is still an open problem. Here the solution of this problem is sought in a combination of the Fourier method with the Duhamel principle in a general sense. The Fourier method is to be understood in a somewhat more general sense than it is accepted in the most textbooks. Also the Duhamel principle is both generalized and extended to space variables. This is done by means of general convolutions of BVP for a class of nonlocal boundary value conditions for linear differential operators of first and second orders. Some numerical experiments, done in the environment of the computer algebra system *Mathematica* are also supplied.

1. Introduction. The Fourier method descends from the celebrated book "Théorie Analitique de la Chaleur" (1822) of Josef Fourier [1]. Till nowadays this method is an indispensable part of that section of applied mathematics which bears the name of mathematical physics. The "explicit" form of the solution given by the method in principle allows to evaluate the solution at each point, independently of the values at other points. This is a great advantage compared with the difference methods, but from the computational point of view it is obtained on a very high price. Indeed, in order to use the solution in a form of a series for evaluation of the values at prescribed points, one should accomplish two time-consuming steps: 1) to expand the boundary value functions into series of eigenfunctions by numerical calculation of many definite integrals (say 50 or more); 2) to form the series solution and to sum it using rather big number of terms (say several hundred) since, as a rule, it is slow convergent.

That's why, if we are looking for a method of solution of boundary value problems of partial differential equations, then both the difference and the Fourier methods are unpractical as it concerns their implementation on personal computers (see Alad'ev and Shishakov [2], p. 644).

Our aim here is to show that this obstacle to an effective computer implementation of the Fourier method on personal computers in most cases could be overcome by a suitable extension of the Duhamel principle. This principle had arisen almost simultaneously with the Fourier method. In 1830 J.-M.-C. Duhamel published a big Memoire [3]. Here, in a rather descriptive form, he had shown that the solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(0,t) = 0,\ u(1,t) = \varphi(t),\ u(x,0) = 0$$

could be obtained for arbitrary $\varphi(t)$, provided that we have the solution $U(x,t)$ of the same problem, but for the special choice $\varphi(t) \equiv 1$. It is given by the formula

$$u(x,t) = \frac{\partial}{\partial t}\int_0^t U(x,t-\tau)\varphi(\tau)d\tau \tag{1}$$

in the strip $0 \le x \le 1,\ 0 \le t$.

Using the Fourier method, we can easily find

$$U(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x \tag{2}$$

It is desirable to extend the Duhamel principle to boundary value problems with non homogenous initial conditions. E.g. for the BVP

$u_t = u_{xx},\ u(0,t) = u(\pi,t) = 0,\ u(x,0) = f(x)$ in the strip $0 \le x \le \pi,\ 0 \le t$

there is also something like the Duhamel representation (1):

$$u(x,t) = \int_0^\pi \left[\theta(x-y,t) - \theta(x+y,t)\right] f(y)dy \tag{3}$$

where

$$\theta(x,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos nx \tag{4}$$

is the well-known θ-function (see Widder [4], p. 94). The function $\theta(x,t)$ is a solution of the heat equation, but it is not a solution of the same BVP for special choice of $f(x)$ since the series for $\theta(x,t)$ diverges for $t=0$. Nevertheless, representation (3) could be used for evaluation of $u(x,t)$ at inner points of the domain $0{\le}x{\le}\pi,\ 0{\le}t$.

Representation (3) dates to 1913 and most likely is due to M. Gevray. As far as we know, no other Duhamel-type representations of this sort for solutions of linear BVP for PDE are known.

Further, we are to show that the Duhamel principle extends to the space variables to the most BVP for PDE for which the Fourier method applies. To this end we are to make a brief survey of the Fourier method in a somewhat more general form than it is usually done in the most textbooks. We aim to encompass nonlocal boundary value problems along with local ones.

2. A survey of the Fourier method, intended for linear boundary value problems. First of all, we are to describe the class of PDE to which our extension of the Fourier method applies. We consider a single PDE of the form

$$\sum_{k=1}^{m} P_k\left(\frac{\partial}{\partial t_k}\right)u = \sum_{l=1}^{n} Q_l\left(\frac{\partial^2}{\partial x_l^2}\right)u + F(x,t), \tag{5}$$

where P_k, $k = 1,\ldots,m$ and Q_l, $l = 1,\ldots,n$ are polynomials with constant coefficients of the corresponding operators and $F(x,t)$ in the right-hand side is a function of $m+n$ variables $(x,t) = (x_1,\ldots,x_n;t_1,\ldots,t_m)$. The domain G in which the solution is sought is supposed to be a Cartesian product of intervals. For the simplicity sake, we assume that

$G = \{(x_1,\ldots,x_n;t_1,\ldots,t_m) : 0 \le x_l \le 1,\ l = 1,\ldots,n;\ 0 \le t_k,\ k = 1,\ldots,m\}$

In order to describe the boundary value conditions for the problems we could treat by the Fourier method, let us suppose that m non-zero linear functionals χ_k, $k = 1, \ldots, m$ in $C[0, \infty)$ and n non-zero linear functionals Φ_l, $l = 1, \ldots, n$ in $C^1[0, 1]$ are given. The functionals χ_k and Φ_l can be given by Stieltjes integrals:

$$\chi_k\{f\} = \int_0^{T_k} f(\tau)d\alpha_k(\tau) \text{ and } \Phi_l\{f\} = \gamma_l f(1) + \int_0^1 f'(\xi)d\beta_l(\xi)$$

with $\gamma_k = \text{const}$, $0 \le T_k \le \infty$, where α_k and β_l are functions with bounded variation.

We consider the equation (5) with the following boundary value conditions:

$$\left.\frac{\partial^{2j}u}{\partial x_l^{2j}}\right|_{x_l=0} = \phi_l^{(j)}(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n; t_1, \ldots, t_m),$$

$$\Phi_l\left\{\frac{\partial^{2j}u}{\partial x_l^{2j}}\right\} = \psi_l^{(j)}(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n; t_1, \ldots, t_m), \tag{6}$$

$$j = 0, 1, 2, \ldots, \deg Q_l - 1,\ l = 1, 2, \ldots, n$$

where the functional Φ_l is applied "partially" to x_l only, and with the initial conditions

$$\chi_k\left\{\frac{\partial^i u}{\partial t_k}\right\} = f_k^{(i)}(x_1, \ldots, x_n; t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_m) \tag{7}$$

$$i = 0, 1, 2, \ldots, \deg P_k - 1,\ k = 1, 2, \ldots, m,$$

where χ is applied partially to t_k only, and $\phi_l^{(j)}, \psi_l^{(j)}$ and $f_k^{(i)}$ are given functions of the indicated variables.

Further, for the simplicity sake, we restrict our considerations to three simplest cases of the equation (5): *a*) the heat equation $u_t = u_{xx} + F(x, t)$; *b*) the wave equation $u_{tt} = u_{xx} + F(x, t)$; *c*) the potential equation $u_{xx} + u_{yy} = F(x, y)$.

In the cases *a*) and *b*) we take the domain G to be the strip $\{0 \le x \le 1, 0 \le t\}$, and in *c*) we take for G the rectangle $0 \le x \le a$, $0 \le y \le b$.

In order BVP (5)-(7) to be solved by the Fourier method we solve the following two elementary BVP:

1. The general spectral problem for the differentiation operator:

$$\frac{dy}{dt} - \mu y = f,\ \chi\{y\} = 0, \tag{8}$$

where χ is an arbitrary linear functional on $C[0, \infty)$. As it is well known, χ has a finite support (by a result of L. Schwartz). Assume supp $\chi \subset [0, T]$. Till 1974 it remained unnoticed the following generalization of the Duhamel convolution

$$(f \overset{(t)}{*} g)(t) = \chi_\tau\left\{\int_\tau^t f(t + \tau - \sigma)g(\sigma)d\sigma\right\} \tag{9}$$

The solution $y = L_\mu f$ determines the resolvent operator L_μ of the spectral problem (8). The values of μ for which (8) has no solution form the spectrum of spectral problem considered. This spectrum is either void (a Cauchy problem), or enumerable (a nonlocal spectral problem). It consists of a sequence of distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_k, \ldots$. Each eigenvalue $\mu_k, k = 1, 2, \ldots$ has its own multiplicity $\varkappa_k$.

To each of these eigenvalues it corresponds a projection operator

$$P_k\{f\} = \frac{1}{2\pi i}\int_{\Gamma_k} L_\mu f d\mu,\ k = 1, 2, \ldots, \tag{10}$$

where Γ_k is a simple contour containing the eigenvalue μ_k only. The operators $P_k, k =$

$1, 2, \ldots$ are called Riesz' projectors on the name of F. Riesz. They play a role similar to that of the finite Fourier transform

$$F_n \{f\} = \int_0^1 f(t) e^{-int} dt,\ n = 0, \pm 1, \pm 2, \ldots \tag{11}$$

in the interval $[0, 1]$. In fact, for $\chi \{f\} = f(1) - f(0)$ we have $P_n \{f\} = F_n \{f\} e^{int}$.

Since L_μ is the convolutional operator $L_\mu = \{e^{\mu t}/E(\mu)\} \ast$, where $E(\mu) = \chi_\tau \{e^{\mu\tau}\}$, i.e. $L_\mu f(t) = \{e^{\mu t}/E(\mu)\} \ast f(t)$, then the Riesz projectors (10) can be considered as the convolution operators

$$P_k \{f\} = \left\{ -\frac{1}{2\pi i} \int_{\Gamma_k} \frac{e^{\mu t}}{E(\mu)} d\mu \right\} \ast f = \varphi_k \ast f \tag{12}$$

too. Here φ_k is a quasi-polynomial of the form $\varphi_k(t) = e^{\mu_k t} \sum_{j=0}^{\varkappa_k - 1} a_{kj} t^j$ where $\varkappa_k$ is the multiplicity of the eigenvalue μ_k.

According to a theorem, due to L. Schwartz and A. F. Leontiev (see Dimovski and Petrova [5]), if supp $\chi = [0, T]$, then the relations $P_k \{f\} = 0, k = 1, 2, \ldots$ imply $f = 0$ on $[0, T]$. In other words, the system of Riesz projectors (10) is total in $C\,[0, T]$.

2. A nonlocal (in general) spectral problem for the square d^2/dx^2 of the differentiation operator d/dx:

$$\frac{d^2 z}{dx^2} + \lambda^2 z = f,\ z(0) = 0,\ \Phi \{z\} = 0, \tag{13}$$

where Φ is an arbitrary linear functional on $C^1\,[0, 1]$ such that $1 \in \operatorname{supp} \Phi$. Till 1976 (see Dimovski [5]) it was not known the following convolution intrinsically connected with the spectral problem (13):

$$(f \overset{(x)}{\ast} g)(x) = -\frac{1}{2} \Phi_\xi \circ \int_0^\xi \left[\int_x^\eta f(\eta + x - \zeta) g(\zeta) d\zeta - \int_{-x}^\eta f_1(\eta - x - \zeta) g_1(\zeta) d\zeta \right] d\eta \tag{14}$$

where $f_1(x) = f(|x|)\operatorname{sign} x$ and $g_1(x) = g(|x|)\operatorname{sign} x$ are the odd continuations of the functions $f(x)$ and $g(x)$. It is a commutative and associative operation in $C\,[0, 1]$, such that the resolvent operator $R_{-\lambda^2}$ determined as the solution $z = R_{-\lambda^2} f$ of the BVP (13) can be represented as the convolution operator

$$R_{-\lambda^2} f = \left\{ \frac{\sin \lambda x}{\lambda G(\lambda)} \right\} \overset{(x)}{\ast} \{f(x)\}, \tag{15}$$

where $G(\lambda) = \Phi_\xi \{\sin \lambda\xi/\xi\}$. The roots of $G(\lambda)$ are the eigenvalues of (13). Let these distinct eigenvalues be $-\lambda_n^2, n = 1, 2, \ldots$ and $\varkappa_n'$ be their respective multiplicities.

The Riesz projectors of (13) are the operators

$$Q_n \{f\} = \frac{1}{\pi i} \int_{\gamma_k'} \lambda R_{-\lambda^2} f d\lambda \tag{16}$$

where γ_k', $k = 1, 2, \ldots$, is a contour containing λ_k only.

In 1989 N. Bozhinov [7] proved that a necessary and sufficient condition for the totality of the Riesz projectors (16) of (13) is the requirement the right endpoint 1 of the segment $[0, 1]$ to belong to the support of the functional Φ, i.e. $1 \in \operatorname{supp} \Phi$. In particulary, Φ could be a functional of the form $\Phi \{f\} = \alpha f'(1) + \beta f(1)$.

The convolution (14) allows the Riesz projectors to be represented as convolution operators of the form

$$Q_n\{f\} = \psi_n * f \text{ with } \psi_n(x) = \int_{\gamma_n'} \frac{\sin\lambda x}{\lambda G(\lambda)} d\lambda, \tag{17}$$

where γ_n' is a small contour around λ_n, which does not contain any other λ_k with $k \neq n$. If $-\lambda_n^2$ is a simple eigenvalue, i.e. if $\varkappa_n = 1$, then $\psi_n(x)$ is the eigenfunction

$$\psi_n(x) = 2\sin\lambda_n x/\lambda_n G'(\lambda_n).$$

Each of the operators Q_n is a projector of $C\,[0,1]$ onto the $\varkappa_n$-dimensional space of the corresponding eigenfunction $\sin\lambda_n x$ and the associated eigenfunctions $x\cos\lambda_n x$, $x^2\sin\lambda_n x$, $x^3\cos\lambda_n x, \dots,$ $x^{2k-1}\cos\lambda_n x$ or $x^{2k}\sin\lambda_n x$ with $[\varkappa_n/2] = k$ in the cases of even or odd $\varkappa_n$, respectively.

Let $f \in C^2\,[0,1]$. Then, looking on $Q_n\{f\} = f_n, n = 1, 2, \dots$ as on an analogon of the finite Fourier transform (see Dimovski and Petrova [6], pp. 94-100), we state the most important operational property of the transform (17).

Theorem 1. *If $f \in C^2\,[0,1]$, then for $f_n = \psi_n * f$ it holds*

$$(f'')_n = (f_n)'' - f(0)\,[(Q_n 1)'' - \psi_n] - \Phi\{f\}\,\psi_n\,. \tag{18}$$

In the special case $\varkappa_n = 1$, i.e. of a simple eigenvalue, we have

$$(f'')_n = -\lambda_n^2 f_n - f(0)\left[(Q_n\{1\})'' - \frac{2\sin\lambda_n x}{\lambda_n G'(\lambda_n)}\right] - \Phi\{f\}\,\frac{2\sin\lambda_n x}{\lambda_n G'(\lambda_n)}. \tag{19}$$

As an inversion formula the identity $f(x) = \sum_{n=1}^{\infty} f_n(x)$ could be used. It is true, when the series is uniformly convergent. If not, then always an Abel-type summation method (see Bozhinov [8]) is available.

The use of transformation (16) or (17) for the solution of the BVP we are looking for, proceeds in the traditional manner. Denote $u_n(x,t) = Q_n u$, $F_n(x,t) = Q_n F$. Then applying (16) on the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(x,t),$$

we get $\partial u_n/\partial t = \partial^2 u_n/\partial x^2 + F_n(x,t)$. Applying (16) on the "initial" condition $\chi_\tau\{u(x,\tau)\} = f(x)$, we obtain $\chi_\tau\{u_n\} = f_n$, where $f_n = Q_n f$. Further we find u_n in a unique way. At last, the solution is written as $u(x,t) = \sum_{i=1}^{\infty} u_n(x,t)$.

Here we can skip the details, since our aim is to propose an alternative of the Fourier method. The only fact, we need further and which can be proven by the generalized Fourier method from this section, is the uniqueness of the solution of BVP (7).

Theorem 2. *If $1 \in \operatorname{supp}\Phi$, then BVP (7) have a unique solution, if and only if $\mu_m + \lambda_n^2 \neq 0$ for all $m, n \in \mathbb{N}$, i.e. iff there is no dispersion relation of the form $\mu_m + \lambda_n^2 = 0$.*

The condition $1 \in \operatorname{supp}\Phi$ ensures the totality of projectors (16). The totality of the projectors (12) is not required.

3. A two-variate operational calculus. Since we aim to avoid the shortcomings of the Fourier method, we should propose a substitute for it. Such a substitute are various operational calculi, intended for boundary value problems. Having on disposal multivariate convolutions, we may use the direct algebraic approach, similar to the approach of J. Mikusinski to the classical Heaviside operational calculus. Nevertheless, a better escape is to use an algebraic construction, based on multipliers fractions, instead of convolution fractions, as it is done by Mikusinski. Here only elements of such an operational calculus are outlined. For more details in a special case, see Chobanov and Dimovski [9].

The starting point is a convolution in $C(\Delta)$ where $\Delta = [0,1] \times [0,\infty)$.

Theorem 3. *Let Φ be a non-zero linear functional on $C^1[0,1]$ and $\tilde{\Phi}$ be the composition of Φ by the integration operator $\int_0^x$, i.e.*

$$\tilde{\Phi}\{f\} = \Phi_\xi \left\{ \int_0^x f(\xi) d\xi \right\} \tag{20}$$

and χ be a linear functional on $C[0,\infty)$. Then the bilinear operation on $C(\Delta)$:

$$(f * g)(x,t) = \chi_\tau \tilde{\Phi}_\xi \{h(x,t;\xi,\tau)\}, \tag{21}$$

with

$$h(x,t;\xi,\tau) = -\frac{1}{2}\int_\tau^t \left[\int_x^\xi f(\xi + x - \eta, t-\tau) g(\eta,\tau) d\eta - \right.$$

$$\left. - \int_{-x}^{\xi} f(|\xi - x - \eta|, t-\tau) g(|\eta|,\tau) \operatorname{sgn}(\xi - x - \eta)\eta d\eta \right] d\tau$$

is a commutative and associative operation in $C(\Delta)$. The resolvent operators L_μ and $R_{-\lambda^2}$ are multipliers of the algebra $[C(\Delta), *]$ and it holds the relation

$$L_\mu R_{-\lambda^2} f = \left\{ \frac{e^{\mu t} \sin \lambda t}{E(\mu)\lambda G(\lambda)} \right\} * f\,. \tag{22}$$

Here we skip the proof, since it follows the lines of a corresponding proof in Chobanov and Dimovski [9].

For the constructing of a corresponding operational calculus, it is convenient to suppose that $\mu = 0$ and $\lambda = 0$ are not eigenvalues of the spectral problems (8) and (13). This is not an essential restriction, but it allows to avoid some technical troubles.

Basic roles in our operational calculus play the operators $l = L_0$ and $L = R_0$. Written explicitly as

$$lu = \int_0^t u(x,\tau)d\tau - \chi_\tau \left\{ \int_0^\tau u(x,\sigma)d\sigma \right\} / \chi\{1\} \tag{23}$$

$$Lu = \int_0^x (x-\xi)u(\xi,t)d\xi - \frac{x}{\Phi\{\xi\}} \Phi_\xi \left\{ \int_0^\xi (\xi - \eta) u(\eta,t) d\eta \right\}, \tag{24}$$

it is immediately seen that they are right inverse operators of the operators $\partial/\partial t$ and $\partial^2/\partial x^2$ respectively.

It is clear also that

$$lu = \{1\} \overset{(t)}{*} u,\ Lu = \{x\} \overset{(x)}{*} u, \tag{25}$$

where $\overset{(t)}{*}$ and $\overset{(x)}{*}$ denote the convolutions (9) and (14), respectively.

Having in our disposal the convolution (21), we consider the space $C(\Delta)$ with the linear operations in it and the operation $*$ as a commutative and associative algebra

$[C(\Delta), *]$ without annihilators. The role of the last condition is explained in Dimovski [5].

The operators l and L have the properties

$l(f * g) = (lf) * g$ and $L(f * g) = (Lf) * g$

Definition. *A linear operator $A : C(\Delta) \to C(\Delta)$ is said to be a multiplier of the convolution algebra $[C(\Delta), *]$, iff the identity $A(f * g) = (Af) * g$ for arbitrary $f, g \in C(\Delta)$ holds.*

All multipliers of the convolution algebra $[C(\Delta), *]$ form a commutative algebra M. Obviously, the algebra M contains a subalgebra, isomorphic to the algebra $[C(\Delta), *]$. It contains also a subalgebra, isomorphic to the field of constants of $C(\Delta)(\mathbb{R} \text{ or } \mathbb{C})$. Hence M can be considered as an algebra on the same field of constants as $[C(\Delta), *]$. After Mikusinski, we say that the constants in M are the numerical operators. If $a \in \mathbb{R}$, then $[a]$ is the numerical operator, determined by a. It acts on a $f \in C(\Delta)$ in the following way: $[a]\, f = \{af(x,t)\}$, i.e. it multiplies f by a. Hence, we may assume that $\mathbb{R} \subset M$. It is convenient also to introduce two other kinds of multipliers.

Definition. *If $\varphi(t) \in C\,[0, \infty)$, then the operator $[\varphi(t)]_x = \varphi \overset{(t)}{*}$ is said to be a numerical operator with respect to x. In a similar way, if $f(x) \in C\,[0,1]$ then by $[f(x)]_t$ we denote the operator $[f(x)]_t = f \overset{(x)}{*}$ and it is called a numerical operator with respect to t.*

Both $[\varphi(t)]_x$ and $[f(x)]_t$ belong to the ring M of the multipliers of the convolution algebra $[C(\Delta), *]$.

Let us denote by N the multiplicative subset of M consisting of the non-zero non-divisors of zero in M.

Due to the isomorphism of $[C(\Delta), *]$ with the subring of M consisting of the operators of the form $f*$, $f \in C(\Delta)$, further we consider $C(\Delta)$ as a part of M, identifying $f*$ and f.

The decisive step is the extension of the ring of M to a bigger ring, the multipliers fractions ring $\mathcal{M}$. In the general algebra courses it is used the denotation $\mathcal{M} = N^{-1}M$. More important is to note that $\mathcal{M}$ consists of fractions of the form A/B with $A \in M$ and $B \in N$. The identity operator of M is also an element of $\mathcal{M}$. We denote it simply by 1.

Basic elements of $\mathcal{M}$ from the point of view of our operational calculus are

$$\frac{1}{l} = s,\ \frac{1}{L} = S$$

In a sense, the elements s and S could be considered as algebraic substitutes of the differential operators $\partial/\partial t$ and $\partial^2/\partial x^2$. But even for $u \in C^2(\Delta)$ the expressions su and Su do not coincide with $\partial u/\partial t$ and $\partial^2 u/\partial x^2$.

Theorem 4. *Let $u \in C^2(\Delta)$. Then*

$$\frac{\partial u}{\partial t} = su - [\chi_\tau \{u(x,\tau\}]_t\,, \tag{26}$$

$$\frac{\partial^2 u}{\partial x^2} = Su - S\,\{u(0,t)(1 - \Phi\,\{1\}\,x)\} - [\Phi_\xi\,\{u(\xi,t)\}]_x \tag{27}$$

The identities (26) and (27) are the key for applications of the operational calculus developed to BVP like (7).

4. Operational calculus approach to BVP (5)–(7). By means of (26) and (27) BVP (5)–(7) for the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$ reduces to the single equation

$$(s - S)u = [f(x)]_t \tag{28}$$

in $\mathcal{M}$. It remains only to divide (28) by $s - S$ and to obtain

$$u = \frac{1}{s - S}[f(x)]_t \tag{29}$$

as its solution in $\mathcal{M}$. But this is possible only when the element $s - S$ is not a divisor of zero in $\mathcal{M}$. This follows from the uniqueness theorem for BVP (5)–(7) (Theorem 2).

Solution (29), represented in the form

$$u = S\frac{1}{s - S}\frac{1}{S}[f(x)]_t$$

allows to interpret u as an element of $C(\Delta)$. Since $1/S = [x]_t$, then

$$\frac{1}{s - S}\frac{1}{S} = \frac{1}{s - S}[x]_t$$

is the solution of problem (7) for the special choice $f(x) = x$. Denoting this solution by $\Omega(x, t)$, solution (29) obtains the Duhamel-type representation

$$u(x, t) = \frac{\partial^2}{\partial x^2}\left\{\Omega(x, t) \overset{(x)}{*} f(x)\right\} \tag{30}$$

Initially, we supposed that $\Phi \in (C^1[0, 1])^*$. But if we restrict our considerations to functionals Φ on $C[0, 1]$ only, then formula (30) could be simplified further.

Theorem 5. *If $\Phi \in (C[0, 1])^*$ and $\Omega(x, t)$ is the solution of BVP (7) for $f(x) = x$, then solution (29) can be represented in the form*

$$u(x, t) = -\frac{1}{2}\frac{\partial}{\partial x}\Phi_\xi\left\{\int_x^\xi \Omega(\xi + x - \eta, t)f(\eta)d\eta + \right. \tag{31}$$

$$\left. + \int_{-x}^\xi \Omega(|\xi - x - \eta|, t)f(|\eta|)\mathrm{sgn}(\xi - x - \eta)\eta d\eta\right\}$$

.

Example 1. The "Samarskii–Ionkin problem" (see Dimovski, Spiridonova [10]). It has the form:

$u_t = u_{xx},\ u(0, t) = 0,\ \int_0^1 u(x, \tau)d\tau = 0,\ u(x, 0) = f(x)$

We have BVP (7) with $\Phi\{f\} = \int_0^1 f(\xi)d\xi$. Then

$$\Omega(x, t) = \sum_{n=1}^{\infty}\{-2x\cos 2n\pi x + 8\pi nt\sin 2n\pi x\}e^{-4n^2\pi^2 t}$$

and

$$u(x, t) = -2\int_0^x \Omega(x - \xi, t)f(\xi)d\xi - \int_x^1 \Omega(1 + x - \xi, t)f(\xi)d\xi \tag{32}$$

$$+\int_{-x}^{1}\Omega(1-x-\xi,t)f(|\xi|)\mathrm{sgn}\xi d\xi,$$

This representation of $u(x,t)$ is very convenient for computer implementation. A visualization of the numerical solution computed by means of the computer algebra system *Mathematica* [12] is shown on Fig. 1. It is preceded by the graph of the function $f(x)$.

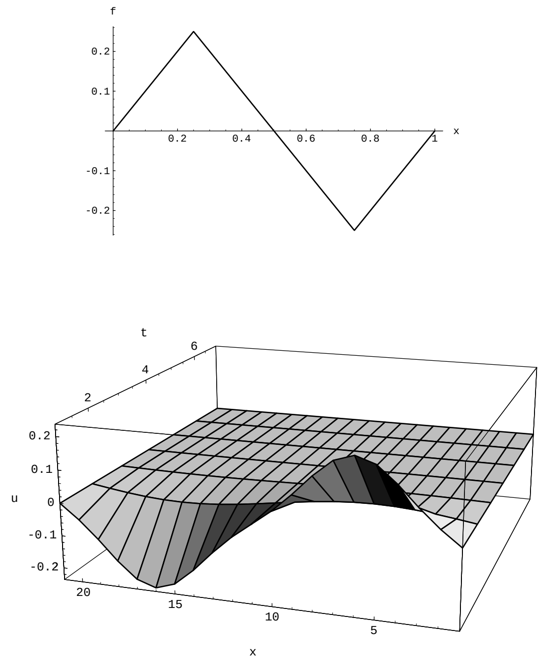

Fig. 1

Now we will consider the possibility to use similar representations for equations of higher order.

Example 2. Consider the equation of a free supported beam (see Farlow [11]):

$$\frac{\partial^2 u}{\partial t^2}=-\frac{\partial^4 u}{\partial x^4},\ 0<x<1,\ 0<t<\infty,$$

with the initial-boundary value conditions:

$$u(0,t)=0,\ u_{xx}(0,t)=0,\ u(1,t)=0,\ u_{xx}(1,t)=0$$

$$u(x,0)=f(x),\ u_t(x,0)=g(x)$$

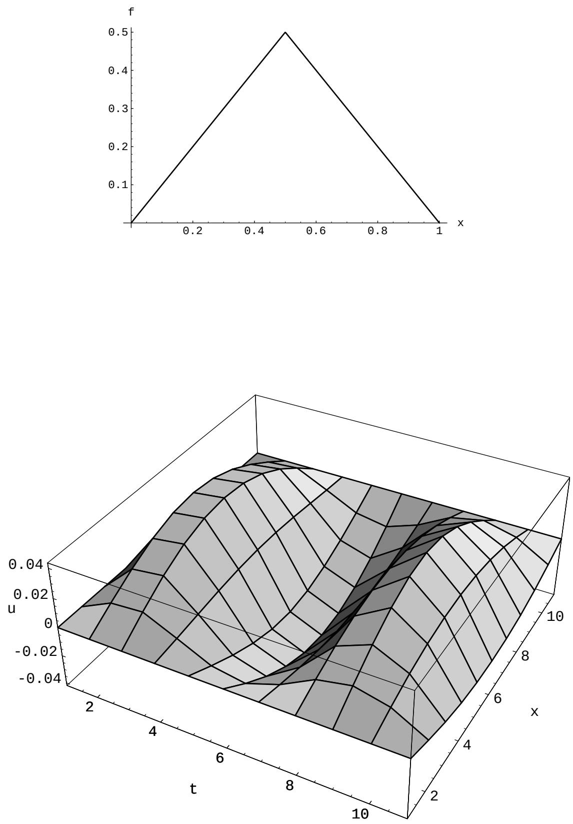

Fig. 2

Using the series solution in Farlow [11] we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin(n\pi)^2 t + b_n \cos(n\pi)^2 t\right] \sin n\pi x \tag{33}$$

where

$$a_n = \frac{2}{(n\pi)^2} \int_0^1 g(x) \sin(n\pi x) dx, \ b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx, \ n = 1, 2, 3, \ldots$$

Further, we consider the case when $f(x) \equiv 0$. Then from the simplified representation (31), we obtain

$$u(x,t) = -\frac{1}{2} \int_x^1 \Omega_x(1 + x - \xi, t) g(\xi) d\xi + \frac{1}{2} \int_{-x}^1 \Omega_x(1 - x - \xi, t) g(|\xi|) \mathrm{sgn}\xi d\xi \tag{34}$$

where

$$\Omega_x(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^{n-1}/n^2)\sin(n\pi)^2 t\cos n\pi x$$

Representation (34) is convenient for computer calculation of the solution. On Fig. 2 the relief of the solution is shown, together with the graph of the chosen function $g(x)$.

The computations and the solution visualization are made in the environment of the computer algebra system *Mathematica* again. A comparison of the numerical solution of the problem with the exact solution was made. Accuracy of order 10^{-7} was achieved when the series for Ω_x is truncated at the 20^{th} term.

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МЕТОДЪТ НА ФУРИЕ И ПРОБЛЕМЪТ ЗА НЕГОВАТА КОМПЮТЪРНА РЕАЛИЗАЦИЯ

Иван Димовски, Маргарита Спиридонова

Компютърната реализация на метода на Фурие за решаване на гранични задачи за ЧДУ е все още открит проблем. Тук решението на този проблем се търси в съчетаването на метода на Фурие с принципа на Дюамел в обобщен смисъл. В основата на предложеното разширение на принципа на Дюамел и за пространствени променливи са общите конволюции за широк клас гранични задачи за линейни диференциални оператори от I и II ред. Извършени са числени експерименти с помощта на системата за компютърна алгебра Mathematica.