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ON A SHORT ANALYTIC PROOF OF THE PRIME NUMBER THEOREM

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The first proofs of the Prime Number Theorem, given in 1896, were too long and used difficult analytic tools. In 1980 D. J. Newman proposed a new proof, simplified later by D. Zagier. Here following the paper of D. Zagier "Newman's short proof of the Prime Number Theorem" [7] we make all its argument clear for the level of interested in analysis undergraduate students. Wide information about the Prime Number Theorem and its proofs could be found in the papers of P. T. Bateman and H. C. Diamond "A hundred years of prime numbers" [1] and J. Korevaar "A century of complex Tauberian Theory" [3]. For interested readers it would be usefull to look at the books of K. Chandrasekharan [2] and E. C. Titchmarsh [6].

1. Preliminaries. The Prime Number Theorem is the following assertion: $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$ where $\pi(x)$ is the number of primes $\leq x$. Recall that the notation $f(x) \sim g(x)$ ("f and g are asymptotically equal") means that $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$. The proof will be present in a series of steps. A sequence of properties of the three functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Phi(s) = \sum_p \frac{\log p}{p^s}, \qquad \theta(x) = \sum_{p\leq x} \log p \qquad (s \in \mathbf{C},\ x \in \mathbf{R}).$$

will be proved. We always use p to denote a prime. The series defining $\zeta(s)$ (the Riemann zeta-function) and $\Phi(s)$ are easily seen to be absolutely and locally uniformly convergent for $\Re(s) > 1$, so they define holomorphic functions in this domain. The Chebyshev function $\theta(x)$ is a piecewise constant, monotonical increasing function with jumps at the prime numbers: $\theta(p_k + 0) - \theta(p_k - 0) = \log p_k$, where p_k is the k-th prime number in the sequence of the prime numbers arranged by increasing.

Needed information from real and complex analysis could be found in the books of G. M. Fihtengoltz [8], B. V. Shabat [9], E. C. Titchmarsh [5] and other books on the topic. The definition of analytic (holomorphic) function would be found in [9], p. 32, the notion of an analytic continuation would be found in [9], Ch. II or in [5], Ch. IV.

(I). Representation of the zeta function as an infinite product (*For a definition of infinite product see for example* [8], *v. II, p.* 350 *or* [5], *Ch. I.*)*:*
$\zeta(s) = \prod\limits_p (1 - p^{-s})^{-1}$ *for* $\Re(s) > 1$.

Proof. From unique factorization each natural number has a representation of the form $n = 2^{r_2}3^{r_3}\ldots$. Then by the absolute convergence of the series representing $\zeta(s)$ we have $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s} = \sum_{r_2,r_3,\ldots\geq 0} (2^{r_2}3^{r_3}\ldots)^{-s} = \prod_p \left(\sum_{r=0}^{\infty} p^{-rs}\right)$ $= \prod_p \frac{1}{1-p^{-s}}$ $(\Re(s) > 1)$. The forth equality holds as the product $\prod_p \left(\sum_{r=0}^{\infty} p^{-rs}\right)$ $= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right)$ contains all the terms of $\sum_{r_2,r_3,\ldots\geq 0} (2^{r_2}3^{r_3}\ldots)^{-s}$ only once. The last equality follows from the formula for the sum of the geometric series with quotient p^{-s}, which is convergent as $|p^{-s}| = p^{-\Re(s)} < 1$ when $\Re(s) > 1$. $\square$

(II). Singularities of the zeta function for $\Re(s) > 0$*:* $\zeta(s) - \frac{1}{s-1}$ *extends analytically to the open half plane* $\Re(s) > 0$*.*

Proof. For $\Re(s) > 1$ holds $\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^s} \int_n^{n+1} dx$ $- \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^s}\,dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx$. The series in the right-hand side converges absolutely for $\Re(s) > 0$ because

$$\left|\int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx\right| = \left|s \int_n^{n+1} \int_n^x \frac{du}{u^{s+1}}\,dx\right| \leq \max_{n\leq u\leq n+1} \left|\frac{s}{u^{s+1}}\right| = \frac{|s|}{n^{\Re(s)+1}}. \qquad \square$$

(III.) Inequality for the Chebyshev function $\theta(x)$*:* $\theta(x) < (4\log 2)x$ *for* $x > 1$*.*

Proof. For $n \in \mathbf{N}$ we obtain $2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \ldots + \binom{2n}{2n} \geq \binom{2n}{n} =$

$$= \frac{2n.(2n-1).\ldots.(n+1)}{1.2.\ldots.n} \geq \prod_{n<p\leq 2n} p = e^{\log \prod_{n<p\leq 2n} p} = e^{\sum_{n<p\leq 2n} \log p} =$$

$= e^{\left(\sum_{p\leq 2n} \log p - \sum_{p\leq n} \log p\right)} = e^{\theta(2n)-\theta(n)}$, i.e. $e^{\theta(2n)-\theta(n)} \leq 2^{2n}$, where from $\theta(2n) - \theta(n) \leq \log 2^{2n} = 2n\log 2$. Let us note that the inequality in the second row above is fulfilled as the product in the right hand-side has as multiples prime numbers which divide the integers in the numerator but not the denominator of the fraction in the left hand-side. Then summing the inequalities $\theta(2n) - \theta(n) \leq 2n\log 2$ for $n = 1, 2, 2^2, \ldots, 2^{m-1}$, we obtain: $\theta(2^m) - \theta(1) = \sum_{r=1}^{m} \left(\theta(2^r) - \theta(2^{r-1})\right) < \log 2 \sum_{r=1}^{m} 2^r < 2^{m+1}\log 2$, and as $\theta(1) = 0$, $\theta(2^m) < 2^{m+1}\log 2$.

Let now $x > 1$ and m be a positive integer such that $2^{m-1} \leq x < 2^m$. As $\theta(x)$ is an increasing function from the last inequality it follows

$$\theta(x) \leq \theta(2^m) < 2^{m+1}\log 2 \leq 4x\log 2.$$

From here $\theta(x) < 4x\log 2$ for $x > 1$, what we are to prove. $\square$

(IV). Nonvanishing of the zeta function and analyticity of $\Phi(s) - \frac{1}{s-1}$ **for** $\Re(s) \geq 1$*:* $\zeta(s) \neq 0$ *and* $\Phi(s) - \frac{1}{s-1}$ *is holomorphic for* $\Re(s) \geq 1$*.*

Proof. For $\Re(s) > 1$, the convergence of the product in **(I)** implies that $\zeta(s) \neq 0$ and

$$-\log\zeta(s) = -\log\prod_p (1-p^{-s})^{-1} = -\sum_p \log(1-p^{-s})^{-1} = \sum_p \log(1-p^{-s})$$

from where it follows that the first derivatives of the first and the last terms are equal. This means

$$-\frac{\zeta'(s)}{\zeta(s)} = \{-\log\zeta(s)\}' = \left\{\sum_p \log\left(1-p^{-s}\right)\right\}' = \sum_p \frac{1}{1-p^{-s}}\left\{1-p^{-s}\right\}' = \sum_p \frac{-(p^{-s})'}{1-p^{-s}}$$

$$= \sum_p \frac{\log p}{p^s-1} = \sum_p \frac{\log p(p^s-1+1)}{p^s(p^s-1)} = \sum_p \frac{\log p}{p^s} + \sum_p \frac{\log p}{p^s(p^s-1)} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}.$$

The final sum converges for $\Re(s) > \frac{1}{2}$, so this fact and **(II)** imply that $\Phi(s)$ is equal to a meromorphic function on $\Re(s) > \frac{1}{2}$, with poles only at $s = 1$ and at the zeros of $\zeta(s)$. In fact, if $\zeta(s)$ has a zero of order μ at s_0, $\Re s_0 > \frac{1}{2}$ then $\zeta(s) = (s-s_0)^\mu A(s)$, $A(s_0) \neq 0$, A being analytic function near s_0. Thus, $-\dfrac{\zeta'(s)}{\zeta(s)} = -\dfrac{\mu}{s-s_0} + \dfrac{A'(s)}{A(s)}$. As $\zeta(s)$ has a pole at $s = 1$ with a residue equal to 1 then $\zeta(s) = \dfrac{A_1(s)}{s-1}$ near $s = 1$, $A_1(s) = 1 + (s-1)B(s)$ with analytic function $B(s)$ near $s = 1$. So we conclude that $-\dfrac{\zeta'}{\zeta}(s)$ has a pole at $s = 1$ with a residue equal to 1. Let now $\zeta(s)$ have a zero of order μ at $s = 1 + i\alpha$ ($\alpha \in \mathbf{R}$, $\alpha \neq 0$) and a zero of order ν at $1 + 2i\alpha$ (so $\mu, \nu \geq 0$ by **(II)**). We can compute easily the following limits:

$$(*) \quad \lim_{\varepsilon\searrow 0} \varepsilon\Phi(1+\varepsilon) = 1, \ \lim_{\varepsilon\searrow 0} \varepsilon\Phi(1+\varepsilon\pm i\alpha) = -\mu, \ \text{ and } \lim_{\varepsilon\searrow 0} \varepsilon\Phi(1+\varepsilon\pm 2i\alpha) = -\nu.$$

E. g. let us calculate the limit $\lim_{\varepsilon\searrow 0} \varepsilon\Phi(1+\varepsilon\pm i\alpha)$. According to the previous considerations with $s = 1+\varepsilon\pm i\alpha$, $s_0 = 1 \pm i\alpha$ we have: $-\dfrac{\zeta'}{\zeta}(1+\varepsilon\pm i\alpha) = -\dfrac{\mu}{\varepsilon} + \dfrac{A'}{A}(1+\varepsilon\pm i\alpha) = \Phi(1+\varepsilon\pm i\alpha) + H(1+\varepsilon\pm i\alpha)$, $H(s)$ being analytic function for $\Re s > \frac{1}{2}$. So $\lim_{\varepsilon\searrow 0} \varepsilon\Phi(1+\varepsilon\pm i\alpha) = -\mu$.

The following inequality holds:

$$\sum_{r=-2}^{2}\binom{4}{2+r}\Phi(1+\varepsilon+ir\alpha) = \sum_{r=-2}^{2}\binom{4}{2+r}\sum_p \frac{\log p}{p^{1+\varepsilon+ir\alpha}}$$

$$= \sum_p \frac{\log p}{p^{1+\varepsilon}}\left(\sum_{r=-2}^{2}\binom{4}{2+r}p^{-ir\alpha}\right) = \sum_p \frac{\log p}{p^{1+\varepsilon}}\left(p^{2i\alpha} + 4p^{i\alpha} + 6p^{0\cdot i\alpha} + 4p^{-i\alpha} + p^{-2i\alpha}\right)$$

$$= \sum_p \frac{\log p}{p^{1+\varepsilon}}p^{-2i\alpha}\left(p^{4i\alpha} + 4p^{3i\alpha} + 6p^{2i\alpha} + p^{i\alpha} + p^{0\cdot\alpha}\right) = \sum_p \frac{\log p}{p^{1+\varepsilon}}p^{-2i\alpha}\left(1+p^{i\alpha}\right)^4$$

$$= \sum_p \frac{\log p}{p^{1+\varepsilon}}\left[(1+p^{i\alpha})(p^{-i\alpha/2})\right]^4 = \sum_p \frac{\log p}{p^{1+\varepsilon}}\left(p^{-i\alpha/2}+p^{i\alpha/2}\right)^4 \geq 0$$

On the other hand, $\displaystyle\sum_{r=-2}^{2}\binom{4}{2+r}\Phi(1+\varepsilon+ir\alpha) = \Phi(1+\varepsilon-2i\alpha) + 4\Phi(1+\varepsilon-i\alpha) + 6\Phi(1+\varepsilon) + 4\Phi(1+\varepsilon+i\alpha) + \Phi(1+\varepsilon+2i\alpha)$. Then applying the limits $(*)$, we obtain that $6 - 8\mu - 2\nu \geq 0$, so $\mu = 0$, i.e., $\zeta(1+i\alpha) \neq 0$. In this way we conclude that for each $R > 0$ there exists $\delta(R) > 0$ such that $\zeta(s)$ is an analytic nonvanishing function in the rectangle $1 - \delta \le \Re s \le 1$, $|\Im s| \le R$ except for the point $s = 1$ where it has a simple pole with residue equal to 1. Therefore, the same is true for $\Phi(s)$. The assertion **(IV)** is verified. $\square$

(V). Convergence of an auxiliary integral: $\displaystyle\int_1^\infty \frac{\theta(x) - x}{x^2}\,dx$ *is a convergent integral.*

Proof. For $\Re(s) > 1$ it is convenient first to consider the Stieltjes integral $\displaystyle\int_1^\infty \frac{d\theta(x)}{x^s}$ (see for example [8], v. III, p. 89). As it is known from the courses on real analysis (see [8], v. III, p. 103):

$$\int_1^\infty \frac{d\theta(x)}{x^s} = \sum_{k=1}^\infty \frac{\theta(p_k + 0) - \theta(p_k - 0)}{p_k^s} = \sum_{k=1}^\infty \frac{\log p_k}{p_k^s} = \Phi(s)$$

where $p_1 = 2, p_2 = 3, \ldots$ are the prime numbers arranged in an increasing order. But after integrating by parts we obtain

$$\int_1^\infty \frac{d\theta(x)}{x^s} = s\int_1^\infty \frac{\theta(x)}{x^{s+1}}\,dx = s\int_0^\infty e^{-st}\theta(e^t)\,dt.$$

So $\Phi(s) = s\displaystyle\int_0^\infty e^{-st}\theta(e^t)\,dt$. Therefore **(V)** is obtained by applying the following Analytic Theorem to the functions $f(t) = \theta(e^t)e^{-t} - 1$ and $g(z) = \dfrac{\Phi(z+1)}{z+1} - \dfrac{1}{z}$, which satisfy its hypothesis according to **(III)** and **(IV)**. Let us make the substitution $s = z+1$ in **(IV)**. Then $\Re s > 1 \iff \Re z > 0$ and $\Phi(z+1) - \dfrac{1}{z}$ is holomorphic for $\Re z \ge 0$. Therefore, the function $g(z) = \dfrac{\Phi(z+1)}{z+1} - \dfrac{1}{z} = \dfrac{1}{z+1}\left(\Phi(z+1) - \dfrac{1}{z} - 1\right)$ is holomorphic for $\Re z \ge 0$. $\square$

Analytic Theorem (of Tauberian type, Neuman [4]). *Let $f(t)$ $(t \ge 0)$ be a bounded and locally integrable function and suppose that the function* $g(z) = \displaystyle\int_0^\infty f(t)e^{-zt}dt$ $(\Re(z) > 0)$ *extends holomorphically to $\Re z \ge 0$. Then* $\displaystyle\int_0^\infty f(t)\quad dt$ *exists (and equals $g(0)$).*

(VI). Asymptotical behaviour of the Chebyshev function $\theta(x)$*:* $\theta(x) \sim x$*.*

Proof. Assume that for some fixed $\lambda > 1$ there are arbitrary large x with $\theta(x) \ge \lambda x$. Since θ is non-decreasing, then for $x \le t \le \lambda x \Rightarrow \theta(t) \ge \theta(x) \ge \lambda x$ and we have

$$\int_x^{\lambda x} \frac{\theta(t) - t}{t^2}\,dt \ge \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt = \int_1^\lambda \frac{\lambda - t}{t^2}\,dt = C(\lambda) = \lambda - 1 - \log\lambda > 0$$

for such x, contradicting **(V)**. Thus, $\overline{\lim}_{x\to\infty}\dfrac{\theta(x)}{x} \le \lambda$, $\forall\lambda > 1$. So $\overline{\lim}_{x\to\infty}\dfrac{\theta(x)}{x} \le 1$.

Similarly, the inequality $\theta(x) \le \lambda x$ with $0 < \lambda < 1$ would imply

$$\int_{\lambda x}^{x} \frac{\theta(t) - t}{t^2}\,dt \le \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2}\,dt = \int_\lambda^1 \frac{\lambda - t}{t^2}\,dt = C(\lambda) = -\lambda + 1 + \log\lambda < 0.$$

We get again a contradiction for λ fixed and x big enough, hence $\displaystyle\underline{\lim}_{x\to\infty}\frac{\theta(x)}{x} \ge 1$. $\square$

Proof of the Prime Number Theorem. The Prime Number Theorem follows easily from **(VI)**, since for any $\varepsilon > 0$ $\theta(x) = \sum_{p \le x} \log p \le \sum_{p\le x} \log x = \pi(x)\log x$, and $\theta(x) \ge \sum_{x^{1-\varepsilon}<p\le x} \log p \ge \sum_{x^{1-\varepsilon}<p\le x} (1-\varepsilon)\log x = [(1-\varepsilon)\log x]\left[\pi(x) - \pi(x^{1-\varepsilon})\right] \ge [(1-\varepsilon)\log x][\pi(x) - x^{1-\varepsilon}]$. Indeed, $\pi(x) \ge \dfrac{\theta(x)}{\log x} \Rightarrow \underline{\lim}_{x\to\infty} \dfrac{\pi(x)}{\frac{x}{\log x}} \ge \lim_{x\to\infty} \dfrac{\theta(x)}{x} = 1$, while $\dfrac{\theta(x)}{x} \ge (1-\varepsilon)\dfrac{\log x}{x}\left[\pi(x) - x^{1-\varepsilon}\right] \Rightarrow 1 = \lim_{x\to\infty}\dfrac{\theta(x)}{x} \ge (1-\varepsilon)\overline{\lim}_{x\to\infty}\dfrac{\pi(x)}{\frac{x}{\log x}}$.

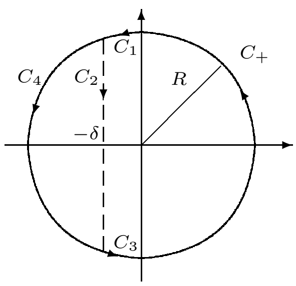
Letting $\varepsilon \to 0$ we obtain the desired result. $\square$

Proof of the Analytic Theorem. For $T > 0$ set $g_T(z) = \int_0^T f(t)e^{-zt}\,dt$. This is clearlya holomorphic function for all z. We must show that $\lim_{T\to\infty} g_T(0) = g(0)$.

Let R be large and fixed and let C be the boundary of the region $\{z \in \mathbf{C} : |z| \le R, \Re(z) \ge -\delta\}$, where $\delta > 0$ is small enough (depending on R) so that $g(z)$ is holomorphic in the region and on C. Then the Cauchy theorem (cf. [9], p. 128 or [5], Ch. II) can be applied to the holomorphic on the domain with boundary C function $h(z) = (g(z) - g_T(z))e^{zT}\left(1 + \dfrac{z^2}{R^2}\right)$ (the origin 0 is contained in the domain). Indeed, by the Cauchy theorem the following equalities hold:

$$(**)\quad g(0)-g_T(0) = h(0) = \frac{1}{2\pi i}\int_C h(z)\,\frac{dz}{z} = \frac{1}{2\pi i}\int_C (g(z)-g_T(z))e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}.$$

Let us consider now the following parts of the curve C and estimate the last integral in (**) on them:



$$C = C_+ \cup C_1 \cup C_2 \cup C_3$$

$$C' = C_+ \cup C_1 \cup C_4 \cup C_3$$

On the semicircle $C_+ = C \cap \{\Re(z) > 0\}$ the integrand of (**) is bounded by $2\frac{B}{R^2}$, where $B = \max_{t\ge 0}|f(t)|$, because

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\,dt\right| \le B\int_T^\infty |e^{-zt}|\,dt = \frac{Be^{-\Re(z)T}}{\Re(z)} \quad (\Re(z) > 0)$$

and $\left|e^{zT}\left(1+\dfrac{z^2}{R^2}\right)\dfrac{1}{z}\right| = \left|e^{\Re(z)T}e^{i\Im(z)T}\right|\left|\dfrac{R^2+z^2}{R^2}\dfrac{1}{z}\right| = e^{\Re(z)T}\left|\dfrac{z\overline{z}+z^2}{zR^2}\right|$

$= e^{\Re(z)T}\left|\dfrac{\overline{z}+z}{R^2}\right| = e^{\Re(z)T}\left|\dfrac{2\Re(z)}{R^2}\right| = e^{\Re(z)T}\dfrac{2\Re(z)}{R^2}$.

Hence the contribution to $g(0) - g_T(0)$ from the integral over C_+ is bounded in absolute value by $\frac{B}{R}$. For the integral over $C_- = C \cap \{\Re(z) < 0\}$ we look at $g(z)$ and $g_T(z)$ separately. Let us denote $C'_- = C' \cap \{\Re(z) < 0\}$.

Since g_T is entire, we have:

$$\int_{C_1 \cup C_2 \cup C_3} g_T(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = \int_{C'_-} g_T(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

as $\displaystyle\int_{C_2} g_T(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z} = \int_{C_4} g_T(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$.

So the integral over C'_- is bounded in absolute value by $2\pi \frac{B}{R}$ by exactly the same estimate as before since

$$|g_T(z)| = \left| \int_0^T f(t) e^{-zt}\, dt \right| \leq B \int_0^T |e^{-zt}|\, dt \leq \frac{B e^{-\Re(z)T}}{|\Re(z)|} \quad (\Re(z) < 0).$$

Fix $R > 0$. Then the remaining integral over C_- tends to 0 according to Lebesgue dominated convergence theorem (cf. [5], Chapter X, 10.5). In fact, $|e^{zT}| \leq 1$ for $\Re z \leq 0$ and $|e^{zT}| \to 0$ for $T \to \infty$ when $\Re z < 0$. Since the set $\{z : \Re z = 0, z \in C_-\} = \{\pm iR\}$ has a mesure 0, then everything is settled. Hence $\overline{\lim}_{T \to \infty} |g(0) - g_T(0)| \leq \frac{2B}{R}$. Since $R > 0$ is arbitrary we conclude that $\displaystyle\int_0^\infty f(t)\, dt$ exists and equals to $g(0)$, what we wanted to prove. $\square$

Corollary of the Prime Number Theorem. *If $\{p_n\}$ is the incresing sequence of the prime numbers, then $p_n \sim n \log n$.*

Proof. Indeed, $\pi(x) \sim \dfrac{x}{\log x}$ means that $\displaystyle\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} = \lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1$. Hence

$$\lim_{x \to \infty} (\log \pi(x) + \log\log x - \log x) = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{\log \pi(x)}{\log x}$$

$$= \lim_{x \to \infty} \frac{\log \pi(x) + \log\log x - \log x}{\log x} + \lim_{x \to \infty} \frac{\log x}{\log x} - \lim_{x \to \infty} \frac{\log\log x}{\log x} = 0 + 1 - 0 = 1.$$

Multiplying the first and the third limits above, we obtain $\displaystyle\lim_{x \to \infty} \frac{\pi(x) \log \pi(x)}{x} = 1$. Putting here $x = p_n$, as it is fulfilled $\pi(x) = n$, we obtain $\displaystyle\lim_{n \to \infty} \frac{n \log n}{p_n} = 1$, i.e. $p_n \sim n \log n$. Let us note also that the assertion in this corollary is equivalent to the Prime Number Theorem (c.f. [2]). $\square$

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REFERENCES

[1] P. T. Bateman, H. C. Diamond. A hundred years of prime numbers. *Amer. Math. Monthly*, **103** (1996), 729–744.

[2] K. Chandrasekharan. Introduction to Analytic Number Theory. Die Grundlehren der Mathematischen Wisenschaften, 148. New York – Berlin – Heidelberg, Springer-Verlag, 1968.

[3] J. Korevaar. A century of complex Tauberian Theory. *Bull. Amer. Math. Soc.*, **39**, No 4 (2002), 475–531.
[4] D. J. Newman. Simple analytic proof of the Prime Number Theorem. *Amer. Math. Monthly*, **87** (1980), 693–696.
[5] E. C. Titchmarsh. The Theory of Functions. Oxford, Oxford Univ. Press, 1939.
[6] E. C. Titchmarsh. The Theory of the Riemann Zeta Function. Oxford, Oxford Univ. Press, 1951.
[7] D. Zagier. Newman's short proof of the Prime Number Theorem. *Amer. Math. Monthly*, **104** (1997), 705–708.
[8] G. M. Fihtengoltz. Kurs diferential'nogo i integral'nogo ischislenija. Moscow, Nauka, 1969.
[9] B. V. Shabat. Vvedenie v kompleksnyi analis. Moscow, Nauka, 1969.

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ЗА ЕДНО КРАТКО АНАЛИТИЧНО ДОКАЗАТЕЛСТВОТО НА ТЕОРЕМАТА ЗА ПРОСТИТЕ ЧИСЛА

Лилия Николова Апостолова, Калин Петров Павлов

Първите доказателства на теоремата за простите числа са дадени през 1896 година. Те са много дълги и използват тежък аналитичен апарат. През 1980 година D. J. Newman дава ново доказателство, опростено по-късно от D. Zagier. В тази статия, следвайки статията на D. Zagier „Кратко доказателство на Нюман на теоремата за простите числа“, (Amer. Math. Monthly, **104** (1997), 705-708), правим изложението достъпно за интересуващи се от математически анализ (знанията, получавани през първите 3 години от обучението на студентите по математика са достатъчни). Обширна информация за теоремата за простите числа може да се намери в статиите на P. Bateman и K. Diamond „Сто години теорема за простите числа“, (Amer. Math. Monthly, **103** (1996), 729-741) и на J. Korevaar “Столетие на комплексната Тауберова теория” (Bull. Amer. Math. Soc., **39** (2002), 475-531). За интересуващите се препоръчваме също книгите на K. Chandarasekhan „Въведение в аналитичната теория на числата“ и на E. C. Titchmarsh „Теория на Римановата Дзета функция“.