МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2003
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2003
Proceedings of the Thirty Second Spring Conference of
the Union of Bulgarian Mathematicians
Sunny Beach, April 5–8, 2003

SOME SIMILAR TRIANGLES ASSOCIATED WITH A PURE TRIANGLE IN THE GALILEAN PLANE*

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It is easy to see that the celebrated trisector theorem in Euclidean elementary geometry of Frank Morley (1860-1937) is not true in the Galilean plane Γ_2. In this paper we show that n-sectors of any pure triangle in Γ_2 determine $n-1$ triangles that are similar to the given triangle with respect to the group $H_4{}''$ of the similitudes of the second type.

1. Introduction. In the affine version the Galilean plane Γ_2 is an affine plane with a special direction which may be taken coincident with the y-axis of the basic affine coordinate system Oxy [3; p.10], [5], [6]. The affine transformations leaving invariant the special direction Oy can be written in the form

$$(1)\qquad \begin{aligned} \overline{x} &= a_1 + a_2 x, \qquad a_2 a_5 \neq 0 \\ \overline{y} &= a_3 + a_4 x + a_5 y, \end{aligned}$$

where $a_1, \ldots, a_5$ are real parameters.

In Γ_2 a straight line is said to be isotropic (or special) if it is parallel to the special direction.

Two points P_1 and P_2 are called parallel if the straight line P_1P_2 is isotropic. Note that the transformations (1) transform the family of isotropic lines of Γ_2 into itself.

The (oriented) distance from the point $P_1(x_1, y_1)$ to the point $P_2(x_2, y_2)$ is defined by [3; p.11], [8; p.51]

$$\overline{P_1P_2} = x_2 - x_1.$$

Let $g_1 : y = k_1 x + n_1$ and $g_2 : y = k_2 x + n_2$ be two nonisotropic straight lines. The (oriented) angle from g_1 to g_2 is defined as [3; p.17-18], [8; p.54]

$$\measuredangle(g_1, g_2) = k_2 - k_1.$$

Note that in Γ_2 any direction has unique the special direction as a perpendicular one.

It is easy to verify that the transformations (1) map a line segment into a proportional line segment and an angle into a proportional angle with the coefficients of proportionality a_2 and $a_2{}^{-1} a_5$, respectively. Thus they form the group H_5 of the general similitudes in Γ_2.

The subgroup $H_4{}' \subset H_5$ determined by the conditions $a_2 = a_5 = \lambda \neq 0$ maps the line segments into proportional ones with the coefficient λ of proportionality, while the

*AMS Subject Classification: 51M99

**The second autor was partially supported by Shumen University under Contract 12/14.03.02

angles are mapped into equal ones. ${H_4}'$ is called the group of similitudes of the first type [6], [8; p.66], or the group of equiform transformations in Γ_2 [4].

If we put $a_2 = 1,\ a_5 = \mu \neq 0$, then we obtain the subgroup ${H_4}'' \subset H_5$ which transforms the line segments into equal ones and the angles into proportional ones with the coefficient μ of proportionality. ${H_4}''$ is called the group of similitudes of the second type [6], [8; p.67].

$\triangle ABC$ is called a pure triangle (or admissible triangle) if the vertices A, B and C are nonparallel points.

Let $\triangle ABC$ be a pure triangle and $\measuredangle(AC, AB) = \alpha, \measuredangle(BA, BC) = \beta, \measuredangle(CB, CA) = \gamma$. Under these assumptions the following relations hold [3; p.22, 28], [8; p.65]:

$$\overline{BC} + \overline{CA} + \overline{AB} = 0, \tag{2}$$

$$\alpha + \beta + \gamma = 0, \tag{3}$$

$$\frac{\overline{BC}}{\alpha} = \frac{\overline{CA}}{\beta} = \frac{\overline{AB}}{\gamma}. \tag{4}$$

2. Dual numbers. Any dual number z can be written in the form [7; p.18]

$$z = z_1 + \varepsilon z_2 \tag{5}$$

where z_1 and z_2 are real numbers and $\varepsilon^2 = 0$. The number z_1 is called the modulus of the dual number z and it is denoted by $|z|$, i.e.

$$|z| = z_1. \tag{6}$$

If $x = x_1 + \varepsilon x_2$ and $y = y_1 + \varepsilon y_2$ are two dual numbers, then the following relations hold [7; p.20]

$$x \pm y = (x_1 \pm y_1) + \varepsilon(x_2 \pm y_2), \quad xy = x_1y_1 + \varepsilon(x_1y_2 + x_2y_1),$$
$$\frac{x}{y} = \frac{x_1}{y_1} + \varepsilon\frac{x_2y_1 - x_1y_2}{{y_1}^2},\ y_1 \neq 0,$$

and

$$|-x| = -|x|,\ |x \pm y| = |x| \pm |y|,\ |xy| = |x||y|,\ |\frac{x}{y}| = \frac{|x|}{|y|}. \tag{7}$$

Let $z = z_1 + \varepsilon z_2$ be a dual number and $z_1 \neq 0$. Then the real number

$$\varphi = \frac{z_2}{z_1} \tag{8}$$

is called the argument of the dual number z and it is denoted by $arg\ z$. Using our notations (5), (6) and (8) we can write

$$z = |z|(1 + \varepsilon\varphi),\ \varphi = arg\ z.$$

3. An identification. Following K. Strubecker [4] we can identify the points of Γ_2 with the dual numbers, as we usually identify the points of the Euclidean plane with the complex numbers. In this case if $x = x_1 + \varepsilon x_2$ and $y = y_1 + \varepsilon y_2$ are two dual numbers with $y_1 \neq x_1$, then they determine two nonparallel points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as

$$\overline{P_1P_2} = y_1 - x_1 = |y - x|.$$

Further, let a, b and c be three dual numbers that determine the pure $\triangle abc$. By using the formulas (2) and (4) we get

$$|c-b|+|a-c|+|b-a|=0,\quad \frac{|c-b|}{\alpha}=\frac{|a-c|}{\beta}=\frac{|b-a|}{\gamma}. \tag{9}$$

4. A lemma. Now, we shall state a technical lemma that will be useful in the sequal.

Lemma 1. For any three dual numbers a, b and c determining a pure $\triangle abc$ with angles $\measuredangle(ac, ab)=\alpha$, $\measuredangle(ba, bc)=\beta$ and $\measuredangle(cb, ca)=\gamma$ the following ralation holds

$$\frac{a-c}{a-b}=-\frac{\beta}{\gamma}(1+\varepsilon\alpha). \tag{10}$$

Proof. Since [8; p.281]

$$\alpha = arg\frac{a-c}{a-b},$$

then

$$\frac{a-c}{a-b}=|\frac{a-c}{a-b}|(1+\varepsilon\ arg\frac{a-c}{a-b})=\frac{|a-c|}{|a-b|}(1+\varepsilon\alpha)$$

and using (9) we deduce (10).

5. The main result. Frank Morley (1860-1937) pioneered the notion of so-called trisector theorem in the Euclidean plane geometry around 1900, but he did not publish a proof of this theorem until 1924 when it appeared in [1] (see also [2]).

Theorem (F. Morley). *The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle* (*Fig.* 1).

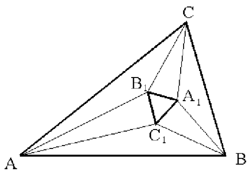

Fig. 1

It is easily seen from (2) and (3) that in Γ_2 there do not exist equilateral triangles and therefore the clasical Morley's trisector theorem is not true.

Now, we shall get another theorem like Morley's by dividing the angles in some other way.

Theorem 1. *Let* $\triangle ABC$ *be a pure triangle in* Γ_2 *and* $\measuredangle(AC, AB)=\alpha$, $\measuredangle(BA, BC)=\beta$, $\measuredangle(CB, CA)=\gamma$. *If*

(i) $p_1, p_2, \ldots, p_{n-1}$ *are straight lines through the vertex A such that* $\measuredangle(AC, p_1)=\measuredangle(p_1, p_2)=\cdots=\measuredangle(p_{n-1}, AB)=\frac{\alpha}{n}$;

(ii) $q_1, q_2, \ldots, q_{n-1}$ *are straight lines through the vertex B such that* $\measuredangle(BA, q_1)=\measuredangle(q_1, q_2)=\cdots=\measuredangle(q_{n-1}, BC)=\frac{\beta}{n}$;

(iii) $r_1, r_2, \ldots, r_{n-1}$ *are straight lines through the vertex C such that* $\measuredangle(CB, r_1)=\measuredangle(r_1, r_2)=\cdots=\measuredangle(r_{n-1}, CA)=\frac{\gamma}{n}$;

(iv) $q_{n-k}\cap r_k = A_k, r_{n-k}\cap p_k = B_k, p_{n-k}\cap q_k = C_k$, *then* $\triangle A_kB_kC_k, k = 1,\dots,n-1$, *are similar to the given* $\triangle ABC$ *with respect to the group* H_4'' *of the similitudes of the second type.*

Proof. Denote by a, b, c, a_k, b_k and c_k the corresponding dual numbers of the points A, B, C, A_k, B_k and C_k, respectively. Since $p_1, p_2, \dots, p_{n-1}$ are nonisotropic straight lines, then the triangle $\triangle A_kBC$ is pure and using Lemma 1, we can write

$$\frac{a_k - c}{a_k - b} = \frac{\beta}{\gamma}(-1 + \varepsilon\frac{k}{n}\alpha)$$

and therefore

$$a_k = \frac{n(b\beta + c\gamma) - \varepsilon kb\alpha\beta}{n(\beta+\gamma) - \varepsilon k\alpha\beta}. \tag{11}$$

Applying (6), (7) and (9) to (11) we find

$$|a_k| = \frac{|n(b\beta + c\gamma) - \varepsilon kb\alpha\beta|}{|n(\beta+\gamma) - \varepsilon k\alpha\beta|} = \frac{|n(b\beta + c\gamma)|}{|n(\beta+\gamma)|} = \frac{|b\beta + c\gamma|}{-\alpha} = -|b|\frac{\beta}{\alpha} - |c|\frac{\gamma}{\alpha} =$$

$$= -|b|\frac{|a-c|}{|c-b|} - |c|\frac{|b-a|}{|c-b|} = \frac{-|b||a| + |b||c| - |c||b| + |c||a|}{|c-b|} = |a|,$$

i.e.

$$|a_k| = |a|. \tag{12}$$

By arguments similar to the ones used above we also establish the equalities

$$|b_k| = |b|,\ |c_k| = |c|. \tag{13}$$

From (12) and (13) it follows

$$|a_k - b_k| = |a - b|,\ |b_k - c_k| = |b - c|,\ |c_k - a_k| = |c - a|$$

and applying (9), we get

$$\frac{\alpha_k}{\alpha} = \frac{\beta_k}{\beta} = \frac{\gamma_k}{\gamma} = \lambda. \tag{14}$$

Formula (14) shows that $\triangle A_kB_kC_k$ can be obtained from $\triangle ABC$ by a similitude of the second type with the coefficient λ of proportionality [8; p.66-67].

From (12) and (13) it follows immediately:

Corollary 1. *The points of n-tuples* $(A, A_1, \dots, A_{n-1})$, $(B, B_1, \dots, B_{n-1})$ *and* $(C, C_1, \dots, C_{n-1})$ *are parallel, respectively.*

In the case $n = 2$ we obtain an interesting characteristic of the bisectors of the angles in any pure triangle.

Corollary 2. *For any pure triangle in* Γ_2 *the three intersection points of the bisectors of the angles in the triangle form a triangle that is similar to the given triangle with respect to the group* H_4'' *of the similitudes of the second type (Fig. 2).*

In the case $n = 3$ we get a Morley like theorem:

Corollary 3. *For any pure* $\triangle ABC$ *in* Γ_2 *the corresponding intersection points* A_1, B_1, C_1 *and* A_2, B_2, C_2 *of the trisectors of the angles in* $\triangle ABC$ *form* $\triangle A_1B_1C_1$ *and* $\triangle A_2B_2C_2$ *that are similar to the given* $\triangle ABC$ *with respect to the group* H_4'' *of the similitudes of the second type (Fig. 3).*

Further, using the duality principle in Γ_2 we can state:

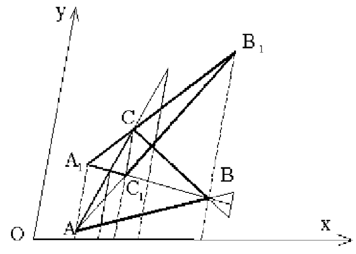

Fig. 2

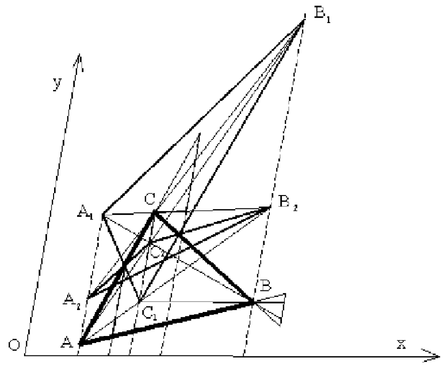

Fig. 3

Theorem 2. *Let $\triangle ABC$ be a pure triangle in Γ_2 and $\overline{BC} = a, \overline{CA} = b, \overline{AB} = c$. If*

(i) $P_1, P_2, \ldots, P_{n-1}$ are points on the side BC such that $\overline{BP_1} = \overline{P_1P_2} = \cdots = \overline{P_{n-1}C} = \frac{a}{n}$;

(ii) $Q_1, Q_2, \ldots, Q_{n-1}$ are points on the side CA such that $\overline{CQ_1} = \overline{Q_1Q_2} = \cdots = \overline{Q_{n-1}A} = \frac{b}{n}$;

(iii) $R_1, R_2, \ldots, R_{n-1}$ are points on the side AB such that $\overline{AR_1} = \overline{R_1R_2} = \cdots = \overline{R_{n-1}B} = \frac{c}{n}$;

(iv) $Q_{n-k}R_k = a_k, \quad R_{n-k}P_k = b_k, \quad P_{n-k}Q_k = c_k$, then $\triangle a_kb_kc_k, k = 1, \ldots, n-1$, are similar to the given $\triangle ABC$ with respect to the group ${H_4}'$ of the similitudes of the first type.

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НЯКОИ ПОДОБНИ ТРИЪГЪЛНИЦИ, СВЪРЗАНИ С ЕДИН ЧИСТ ТРИЪГЪЛНИК В ГАЛИЛЕЕВАТА РАВНИНА

Адриян Върбанов Борисов, Маргарита Георгиева Спирова

Лесно се вижда, че знаменитата теорема на Франк Морли (1860-1937) за трисектрисите в евклидовата елементарна геометрия не е вярна в галилеевата равнина Γ_2. В тази статия показваме, че n-сектрисите на всеки чист (допустим) триъгълник в Γ_2 определят $n-1$ триъгълника, които са подобни на дадения триъгълник относно групата H_4'' на подобностите от втори тип.