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MEASURABILITY OF SETS OF PAIRS OF POINTS IN THE SIMPLY ISOTROPIC SPACE*

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The measurable sets of pairs of points and the corresponding invariant densities with respect to the group of the general similitudes and some its subgroups are given.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is defined [6], [8], [9] as a projective space $\mathbb{P}_3(\mathbb{R})$ in which the absolute consists of a plane ω and two complex conjugate straight lines f_1, f_2 into ω with a (real) intersection point F. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of the general simply isotropic similitudes. Passing on to affine coordinates (x, y, z) a similitude of G_8 can be written in the form [6; p.3]

$$
\begin{array}{rcl}
\overline{x} & = & a + p(x\cos\varphi - y\sin\varphi), \\
\overline{y} & = & b + p(x\sin\varphi + y\cos\varphi), \\
\overline{z} & = & c + c_1x + c_2y + c_3z,
\end{array}
\tag{1}
$$

where $p > 0, \varphi, a, b, c, c_1, c_2$ and c_3 are real parameters.

A straight line is said to be (completely) isotropic if its infinite point coincides with the absolute point F; otherwise the straight line is said to be nonisotropic [6; p.5].

Two points P_1 and P_2 are called parallel if the straight line P_1P_2 is isotropic. The distance $\delta(P_1, P_2)$ between two nonparallel points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined by

$$\delta(P_1, P_2) = +\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{2}$$

If $P_1(x, y, z_1)$ and $P_2(x, y, z_2)$ are parallel points, then the distance $s(P_1, P_2)$ from P_1 to P_2 is given by the real number [6; p.6]

$$s(P_1, P_2) = z_2 - z_1. \tag{3}$$

The δ-distance (2) between two nonparallel points and the s-distance (3) between two parallel points in ${I_3}^{(1)}$ are relative invariants of the group G_8.

We shall consider G_8 and the following its subgroups:

I. $B_7 \subset G_8 \iff p = 1$. It is the group of the simply isotropic similitudes of the δ-distance [6; p.5].

II. $S_7 \subset G_8 \iff c_3 = 1$. It is the group of the simply isotropic similitudes of the s-distance [6; p.6].

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III. $W_7 \subset G_8 \iff c_3 = p$. It is the group of the simply isotropic angular similitudes [6; p.18].

IV. $G_7 \subset G_8 \iff \varphi = 0$. It is the group of the boundary simply isotropic similitudes [6; p.8].

V. $V_7 \subset G_8 \iff c_3 p^2 = 1$. It is the group of the volume preserving simply isotropic similitudes [6: p.8].

VI. $G_6 = G_7 \cap V_7$. It is the group of the volume preserving boundary simply isotropic similitudes [6; p.8].

VII. $B_6 = B_7 \cap G_7$. It is the group of the modular boundary motions [6; p.9].

VIII. ${B_6}^{(1)} = B_7 \cap S_7$. It is the group of the simply isotropic motions [6; p.6].

IX. $B_5 = B_6 \cap {B_6}^{(1)}$. It is the group of the unimodular boundary motions [6; p.9].

We emphasize that much of the common material of the geometry of the simply isotropic space ${I_3}^{(1)}$ can be found in [6], [8] and [9].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [7], G. I. Drinfel'd and A. V. Lucenko [3], [4], [5] we study the measurability of sets of pairs of points in ${I_3}^{(1)}$ with respect to G_8 and the indicated above subgroups. Analogous problems for sets of spheres in ${I_3}^{(1)}$ have been treated in [1].

2. Measurability with respect to G_8. (A) Let $(P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2))$ be a pair of nonparallel points in ${I_3}^{(1)}$, i. e.

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 \neq 0.$$

Unter the action of (1) the pair $(P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2))$ is transformed into the pair $(\overline{P_1}(\overline{x_1}, \overline{y_1}, \overline{z_1}), \overline{P_2}(\overline{x_2}, \overline{y_2}, \overline{z_2}))$ as

$$\begin{array}{rcll} \overline{x_i} & = & a + p(x_i \cos\varphi - y_i \sin\varphi), & \\ \overline{y_i} & = & b + p(x_i \sin\varphi + y_i \cos\varphi), & i = 1, 2, \\ \overline{z_i} & = & c + c_1 x_i + c_2 y_i + c_3 z_i. & \end{array} \tag{4}$$

The transformations (4) form the so-called associated group $\overline{G_8}$ of G_8 [7; 34]. $\overline{G_8}$ is isomorphic to G_8 and the invariant density with respect to G_8 of the pairs (P_1, P_2), if it exists, coincides with the invariant density with respect to $\overline{G_8}$ of the points with coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ in the set of parameters [7; 33]. The group $\overline{G_8}$ has the infinitesimal operators

$$\begin{gathered} Y_1 = \tfrac{\partial}{\partial x_1} + \tfrac{\partial}{\partial x_2}, \quad Y_2 = \tfrac{\partial}{\partial y_1} + \tfrac{\partial}{\partial y_2}, \quad Y_3 = \tfrac{\partial}{\partial z_1} + \tfrac{\partial}{\partial z_2}, \\ Y_4 = x_1 \tfrac{\partial}{\partial x_1} + y_1 \tfrac{\partial}{\partial y_1} + x_2 \tfrac{\partial}{\partial x_2} + y_2 \tfrac{\partial}{\partial y_2}, \quad Y_5 = -y_1 \tfrac{\partial}{\partial x_1} + x_1 \tfrac{\partial}{\partial y_1} - y_2 \tfrac{\partial}{\partial x_2} + x_2 \tfrac{\partial}{\partial y_2}, \\ Y_6 = x_1 \tfrac{\partial}{\partial z_1} + x_2 \tfrac{\partial}{\partial z_2}, \quad Y_7 = y_1 \tfrac{\partial}{\partial z_1} + y_2 \tfrac{\partial}{\partial z_2}, \quad Y_8 = z_1 \tfrac{\partial}{\partial z_1} + z_2 \tfrac{\partial}{\partial z_2}. \end{gathered} \tag{5}$$

We distinguish the following cases:

(i) $x_2 - x_1 \neq 0$. In this case the infinitesimal operators Y_1, Y_2, Y_3, Y_4, Y_5 and Y_6 are arcwise unconnected and

$$Y_8 = \frac{x_2 z_1 - x_1 z_2}{x_2 - x_1} Y_3 + \frac{z_2 - z_1}{x_2 - x_1} Y_6.$$

Since

$$Y_3\left(\frac{x_2 z_1 - x_1 z_2}{x_2 - x_1}\right) + Y_6\left(\frac{z_2 - z_1}{x_2 - x_1}\right) \neq 0,$$

we deduce that a set of pairs of nonparallel points of type (i) is not measurable and it has not measurable subsets with respect to the group G_8.

(ii) $y_2 - y_1 \neq 0$. Now the infinitesimal operators Y_1, Y_2, Y_3, Y_4, Y_5 and Y_7 are arcwise unconnected and

$$Y_8 = \frac{y_2 z_1 - y_1 z_2}{y_2 - y_1} Y_3 + \frac{z_2 - z_1}{y_2 - y_1} Y_7.$$

It is easy to see that

$$Y_3\left(\frac{y_2 z_1 - y_1 z_2}{y_2 - y_1}\right) + Y_7\left(\frac{z_2 - z_1}{y_2 - y_1}\right) \neq 0$$

and therefore a set of pairs of nonparallel points of type (ii) is not measurable and it has not measurable subsets with respect to the group G_8.

(B) Let $(P_1(x, y, z_1), P_2(x, y, z_2))$ be a pair of parallel points, i.e.

$$z_2 - z_1 \neq 0. \tag{6}$$

Now the corresponding associated group $\overline{G_8}$ of G_8 has the infinitesimal operators

$$\begin{aligned} &Y_1 = \tfrac{\partial}{\partial x}, \quad Y_2 = \tfrac{\partial}{\partial y}, \quad Y_3 = \tfrac{\partial}{\partial z_1} + \tfrac{\partial}{\partial z_2}, \\ &Y_4 = xY_1 + yY_2, \quad Y_5 = -yY_1 + xY_2, \quad Y_6 = xY_3, \\ &Y_7 = yY_3, \quad Y_8 = z_1 \tfrac{\partial}{\partial z_1} + z_2 \tfrac{\partial}{\partial z_2}. \end{aligned} \tag{7}$$

From (6) it follows that the infinitesimal operators Y_1, Y_2, Y_3 and Y_8 are arcwise unconnected, but $Y_1(x) + Y_2(y) \neq 0$ and consequently a set of pairs of parallel points is not measurable and has not measurable subsets under the group G_8.

We summarize the foregoing results in the following

Theorem 1. *Sets of pairs of points are not measurable with respect to the group G_8 and have no measurable subsets.*

3. Measurability with respect to B_7. (A) Considering a set of pairs $(P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2))$ of nonparallel points, we obtain that the associated group $\overline{B_7}$ of the group B_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_5, Y_6, Y_7$ and Y_8 from (5). The group $\overline{B_7}$ acts intransitively on the set of points $(x_1, y_1, z_1, x_2, y_2, z_2)$ and therefore the pairs (P_1, P_2) of nonparallel points have not invariant density under B_7. The system

$$Y_1(f) = 0,\ Y_2(f) = 0,\ Y_3(f) = 0,\ Y_5(f) = 0,\ Y_6(f) = 0,\ Y_7(f) = 0,\ Y_8(f) = 0$$

has an integral

$$f = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and it is an absolute invariant of $\overline{B_7}$. Consider the subset of pairs of nonparallel points satisfying the condition

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = h^2, \tag{8}$$

where $h = const \neq 0$. The group $\overline{B_7}$ induces on the subset (8) the group $B_7{}^{\star}$ with the infinitesimal operators

$$Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \quad Z_2 = \frac{\partial}{\partial y_1}, \quad Z_3 = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2},$$

$$Z_5 = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - [y_1 + \sqrt{h^2 - (x_2 - x_1)^2}] \frac{\partial}{\partial x_2},$$

$$Z_6 = x_1 \frac{\partial}{\partial z_1} + x_2 \frac{\partial}{\partial z_2}, \quad Z_7 = y_1 \frac{\partial}{\partial z_1} + [y_1 + \sqrt{h^2 - (x_2 - x_1)^2}] \frac{\partial}{\partial z_2},$$

$$Z_8 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$

It is easy to verify that

$$\lambda Z_3 + \mu Z_7 + \nu Z_8 = 0,$$

where $\lambda = -y_1(z_1 - z_2) - z_1\sqrt{h^2 - (x_2 - x_1)^2}$, $\mu = z_1 - z_2$, $\nu = \sqrt{h^2 - (x_2 - x_1)^2}$. Because on the whole $h^2 - (x_2 - x_1)^2 \neq 0$ we have $Z_3(\lambda) + Z_7(\mu) + Z_8(\nu) \neq 0$ and from here it follows that a set of pairs of nonparallel points satisfying (8) is not measurable with respect to the group B_7 and has not measurable subsets.

(B) Let us consider a set of pairs $(P_1(x, y, z_1), P_2(x, y, z_2))$ of parallel points. The corresponding associated group $\overline{B_7}$ of B_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_5, Y_6, Y_7$ and Y_8 from (7) and it acts transitively on the set of parameters (x, y, z_1, z_2). The integral invariant function $f = f(x, y, z_1, z_2)$ satisfies the so-called system of R. Deltheil [2; p.28], [7; p.11]

$$Y_1(f) = 0, \ Y_2(f) = 0, \ Y_3(f) = 0, \ Y_5(f) = 0, \ Y_8(f) + 2f = 0$$

and has the form

$$f = \frac{h}{(z_2 - z_1)^2},$$

where $h = const.$

From these considerations, we obtain:

Theorem 2. *With respect to the group* B_7

(i) *a set of pairs of nonparallel points is no measurable and has not measurable subsets.*

(ii) *a set of pairs* $(P_1(x, y, z_1), P_2(x, y, z_2))$ *of parallel points is measurable and has the invariant density*

$$d(P_1, P_2) = \frac{1}{s^2} dx \wedge dy \wedge dz_1 \wedge dz_2,$$

where s *is the distance* (3).

4. Measurability with respect to $S_7, W_7, G_7, V_7, G_6, B_6, B_6{}^{(1)}$ and B_5. By arguments similar to the ones used above we examine the measurability of sets of pairs of points with respect to all the rest groups. We collect the results in the following table:

group	a set of pairs of nonparallel points	a set of pairs of parallel points
S_7	$d(P_1, P_2) =$ $= \frac{1}{\delta^4} dx_1 \wedge dy_1 \wedge dz_1 \wedge dx_2 \wedge dy_2 \wedge dz_2$	it is not measurable and has not measurable subsets
W_7	$d(P_1, P_2) =$ $= \frac{1}{\delta^6} dx_1 \wedge dy_1 \wedge dz_1 \wedge dx_2 \wedge dy_2 \wedge dz_2$	$d(P_1, P_2) =$ $= \frac{1}{s^4} dx \wedge dy \wedge dz_1 \wedge dz_2$
G_7	it is not measurable and has no measurable subsets	it is not measurable and has no measurable subsets
V_7	$d(P_1, P_2) =$ $= dx_1 \wedge dy_1 \wedge dz_1 \wedge dx_2 \wedge dy_2 \wedge dz_2$	$d(P_1, P_2) =$ $=\frac{1}{\lvert s \rvert} dx \wedge dy \wedge dz_1 \wedge dz_2$
G_6	it is not measurable but has the measurable subsets $y_2 - y_1 = h_1(x_2 - x_1), x_2 - x_1 \neq 0, h_1 = const$ $x_2 - x_1 = h_2(y_2 - y_1), y_2 - y_1 \neq 0, h_2 = const$ with the densities $d(P_1, P_2) = \lvert x_2 - x_1 \rvert dx_1 \wedge dy_1 \wedge dz_1 \wedge dx_2 \wedge dz_2$, $d(P_1, P_2) = \lvert y_2 - y_1 \rvert dx_1 \wedge dy_1 \wedge dz_1 \wedge dy_2 \wedge dz_2$.	$d(P_1, P_2) =$ $= \frac{1}{\lvert s \rvert} dx \wedge dy \wedge dz_1 \wedge dz_2$
B_6	it is not measurable and has no measurable subsets	$d(P_1, P_2) =$ $= \frac{1}{s^2} dx \wedge dy \wedge dz_1 \wedge dz_2$
$B_6{}^{(1)}$	it is not measurable but has the measurable subset $(x_2 - x_1)^2 + (y_2 - y_1)^2 = h^2, h = const \neq 0$ with the density $d(P_1, P_2) =$ $= \frac{1}{\sqrt{h^2 - (x_2 - x_1)^2}} dx_1 \wedge dy_1 \wedge dz_1 \wedge dx_2 \wedge dz_2$	it is not measurable but has the measurable subsets $z_2 - z_1 = h, h = const \neq 0$ with the density $d(P_1, P_2) = dx \wedge dy \wedge dz_1$
B_5	it is not measurable but has the measurable subset $x_2 - x_1 = h_1, y_2 - y_1 = h_2, h_1, h_2 = const$ $\lvert h_1 \rvert + \lvert h_2 \rvert \neq 0$ with the density $d(P_1, P_2) = dx_1 \wedge dy_1 \wedge dz_1 \wedge dz_2$	it is not measurable but has the measurable subset $z_2 - z_1 = h, h = const \neq 0$ with the density $d(P_1, P_2) = dx \wedge dy \wedge dz_1$

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ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ ТОЧКИ В ПРОСТО ИЗОТРОПНО ПРОСТРАНСТВО

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Намерени са измеримите множества от двойки точки и съответните им инвариантни гъстоти относно общата група на подобностите и някои важни нейни подгрупи.