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ON THE EXISTENCE OF NONTRIVIAL SOLUTIONS OF A NONLINEAR FOURTH ORDER BOUNDARY VALUE PROBLEM *

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A result conserning the existence of infinitely many nontrivial solutions of a nonlinear fourth order boundary value problem is proved. The task is related to the stationary periodic solutions of Fisher–Kolmogorov and Swift–Hohenberg equations. The latter equations describe the phase transition phenomena in a singular Lifschitz point. A symmetric version of the mountain pass theorem and the Hilbert spectral theorem for self-adjoint compact operators are used.

1.Introduction. This paper deals with the existence of nontrivial solutions of the fourth order boundary value problem

$$\text{(1)}\qquad \begin{cases} u^{iv} - pu'' + a(x)\,u - b(x)\,u^3 = 0, & 0 < x < L, \\ u(0) = u''(0) = u(L) = u''(L) = 0, \end{cases}$$

where a and b are positive continuous functions, and p and L are real parameters, $L > 0$. The problem (1) is related to the stationary odd $2L-$periodic solutions of the equation

$$\text{(2)}\qquad \frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4} - p\frac{\partial^2 u}{\partial x^2} + \tilde{a}(x)\,u - \tilde{b}(x)\,u^3,$$

where

$$\tilde{a}(x) := \begin{cases} a(x), & x \in [0, L], \\ a(-x), & x \in (-L, 0), \\ 2L - periodic, & \sim, \end{cases} \qquad \tilde{b}(x) := \begin{cases} b(x), & x \in [0, L], \\ b(-x), & x \in (-L, 0), \\ 2L - periodic, & \sim . \end{cases}$$

Equations of the type of (2) arrise as model equations for phase transitions near a singular Lifschitz point ([4], [10]), also for spatial patterns formation in bistable systems, [3]. Their stationary bounded solutions (periodic, homoklinic and heteroklinic functions) are studied in the works of Peletier & Troy [7], [8], Kalies, Kwapisz & Van der Vorst [6], Van den Berg, Peletier & Troy [1], Habets, Sanchez, Tarallo & Terrachini [5], as well as in Tersian & Chaparova [2], [11], [12].

We put the problem (1) in a variational setting introducing the functional

$$I(u) := \int_0^L \left(\frac{1}{2}\left(u''^2 + pu'^2 + a(x)\,u^2 \right) - \frac{1}{4} b(x)\,u^4 \right) dx$$

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in the Sobolev space $X := H^2(0,L) \cap H_0^1(0,L)$ with the inner product

$$(u,v) := \int_0^L (u''v'' + u'v' + uv)\, dx$$

and the corresponding norm

$$\|u\| := \sqrt{(u,u)}.$$

The problem (1) is considered in case $p > 0$ in Tersian & Chaparova [11]. Using a symmetric version of the mountain pass theorem ([9], Theorem 9.12) it is proved

Theorem 1. *Let $p > 0$, and a and b be positive and continuous functions in $[0,L]$. The problem (1) possesses infinitely many pairs of solutions $(u_n, -u_n)$ which are critical points of the functional I, and $I(u_n) \to +\infty$ as $n \to \infty$.*

In the present paper it is shown that (1) has infinitely many solutions $(u_n, -u_n)$ also for $-2\sqrt{a_1} < p < 0$, where $a_1 := \min_{[0,L]} a(x)$.

2. Existence results. A careful look at the proof of Theorem 1 shows that the assumption $p > 0$ is used only in order the expression

$$|||u||| := \left(\int_0^L \left(u''^2 + pu'^2 + a(x)\, u^2 \right) dx \right)^{1/2} \tag{3}$$

to be an equivalent norm in X. It is natural to ask if (3) is an equivalent norm in X for some $p < 0$. The answer is given in the following

Lemma 1. *Let $-2\sqrt{a_1} < p < 0$ where $a_1 := \min_{[0,L]} a(x)$. Then $|||\cdot|||$, defined by (3) is an equivalent norm in X.*

The proof of Lemma 1 relies on the well known Hilbert spectral theorem for self-adjoint compact operators. We state it for completeness.

Theorem 2. (the Hilbert spectral theorem) *Let $a : H \times H \to \mathbf{R}$ be an equivalent scalar product in the Hilbert space H, $b : H \times H \to \mathbf{R}$ be a continuous symmertic bilinear form such that*

(i) $u_n \to u$ weakly in H implies $b(u_n, u_n) \to b(u,u)$,
(ii) $b(u,u) > 0$ for $u \in H \setminus \{0\}$.

Then the eigenvalue problem

$$a(u,v) = \lambda b(u,v), \quad \forall v \in H$$

has a sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le ...$$

and corresponding sequence of eigenvectors (e_n) such that

(j) $\lambda_n \to +\infty$;
(jj) $\lambda_n = \min_{u \in \{e_1, e_2, \ldots, e_{n-1}\}^\perp} \dfrac{a(u,u)}{b(u,u)}$;
(jjj) $\{e_n\}$ is orthonormal with respect to b and orthogonal with respect to a;
(jv) $\{e_n\}$ is an orthogonal basis in H.

As a consequence of the Theorem 2 we have

Proposition 1. *Suppose a and b are as in the Theorem 2. If*

$$\lambda_1 := \min_{u \in H \setminus \{0\}} \frac{a(u,u)}{b(u,u)} > 1,$$

then $(a(u,u) - b(u,u))^{1/2}$ is an equivalent norm in H.

Proof of Lemma 1. Consider the eigenvalue problem: find $(\lambda, u) \in \mathbf{R} \times X$, $u \neq 0$ such that

$$\int_0^L (u''\phi'' + a(x) u\phi)\, dx = \lambda \int_0^L (-p) u'\phi' dx, \quad \forall \phi \in X. \tag{4}$$

Let (λ, u) be a solution of (4). Since

$$\int_0^L u'\phi' dx = -\int_0^L u''\phi dx, \quad \forall \phi \in X,$$

by (4) it follows that there exists the second weak derivative of u'' belonging to $L^2(0,L)$, and

$$\begin{gathered} u^{iv} + a(x) u = \lambda p u'' \quad a.e. \ in \ (0,L), \\ u(0) = u''(0) = u(L) = u''(L) = 0. \end{gathered} \tag{5}$$

We put

$$x = Lt, \quad 0 < t < 1$$

in (5) and denote

$$v(t) := u(Lt), \quad \bar{a}(t) = a(Lt).$$

Thus (5) has the equivalent form

$$\begin{gathered} \frac{1}{L^4} v^{iv} + \bar{a}(x) v = \lambda p \frac{1}{L^2} v'' \quad a.e. \ in \ (0,1), \\ v(0) = v''(0) = v(1) = v''(1) = 0. \end{gathered} \tag{6}$$

Multiplying the equation in (6) by $\psi \in \bar{X} := H^2(0,1) \cap H_0^1(0,1)$ and integrating it in $(0,1)$, we obtain

$$\int_0^1 \left(\frac{1}{L^4} v''\psi'' + \bar{a}(t) v\psi \right) dt = \lambda \int_0^1 (-p) \frac{1}{L^2} v'\psi' dt, \quad \forall \psi \in \bar{X}. \tag{7}$$

Hence $(\lambda, u) \in \mathbf{R} \times X$ is a solution of (4) if and only if $(\lambda, v(t) := u(Lt)) \in \mathbf{R} \times \bar{X}$ is a solution of (7).

Since

$$\int_0^1 v'^2 dt = -\int_0^1 vv'' dt \leq \frac{1}{2} \int_0^1 v^2 dt + \frac{1}{2} \int_0^1 v''^2 dt$$

for every $v \in \bar{X}$ it follows that

$$\left(\int_0^1 \left(\frac{1}{L^4} v''^2 + \bar{a}(t) v^2 \right) dt \right)^{1/2}$$

is an equivalent norm in $\bar{X}$. The compact embedding of $\bar{X}$ in $C^1[0,1]$ yields that (7) satisfies the hypotheses of the Theorem 2. From (jj), for the first eigenvalue $\lambda_1(L)$ of

(7) we have

$$\lambda_1(L) = \min_{v\in\bar{X}\setminus\{0\}} \frac{\int_0^1 \left(\frac{1}{L^4}v''^2 + \bar{a}(t)\, v^2\right) dt}{\displaystyle\int_0^1 (-p)\, \frac{1}{L^2} v'^2 dt}.$$

The function

$$\varphi_v(L) := \frac{\int_0^1 \left(\frac{1}{L^4}v''^2 + \bar{a}(t)\, v^2\right) dt}{\displaystyle\int_0^1 (-p)\, \frac{1}{L^2} v'^2 dt}$$

is continuously differentiable in $(0,\infty)$, and

$$\varphi_v'(L) = \frac{-\frac{2}{L^3}(-p)\left(\displaystyle\int_0^1 v'^2 dt\right)}{\left(\int_0^1 (-p)\,\frac{1}{L^2}v'^2 dt\right)^2}\left[\frac{1}{L^4}\left(\int_0^1 v''^2 dt\right) - \int_0^1 \bar{a}(t)\, v^2 dt\right].$$

Thus $\varphi_v(L)$ has a global minimum at $L = \bar{L}_v := \left(\left(\int_0^1 v''^2 dt\right) / \left(\int_0^1 \bar{a}(t)\, v^2 dt\right)\right)^{1/4}$, $\varphi_v(L)$ decreases in $\left(0, \bar{L}_v\right)$ and increases in $\left(\bar{L}_v, \infty\right)$, $\lim_{L\to 0}\varphi_v(L) = +\infty$, and $\lim_{L\to\infty}\varphi_v(L) = \int_0^1 \bar{a}(t)\, v^2 dt$. Then

$$\begin{aligned}
\varphi_v(L) &\geq \frac{\int_0^1 \left(\frac{1}{L_v^4}v''^2 + \bar{a}(t)\, v^2\right) dt}{\int_0^1 (-p)\,\frac{1}{L_v^2}v'^2 dt} \\
&= \frac{2\left(\int_0^1 \bar{a}(t)\, v^2 dt\right)^{1/2}\left(\int_0^1 v''^2 dt\right)^{1/2}}{(-p)\int_0^1 v'^2 dt} \\
&> \frac{2\sqrt{a_1}\left(\int_0^1 v^2 dt\right)^{1/2}\left(\int_0^1 v''^2 dt\right)^{1/2}}{2\sqrt{a_1}\int_0^1 v'^2 dt}.
\end{aligned}$$

Since

$$\int_0^1 v'^2 dt = -\int_0^1 vv'' dt \leq \left(\int_0^1 v^2 dt\right)^{1/2}\left(\int_0^1 v''^2 dt\right)^{1/2}$$

for every $v \in \bar{X}$ we deduce that

$$\varphi_v(L) > 1$$

for every $L \in (0,\infty)$ and $v \in \bar{X}\setminus\{0\}$. Hence, for the first eigenvalue $\lambda_1(L)$ of (7) we obtain

$$\lambda_1(L) = \min_{v\in\bar{X}\setminus\{0\}} \varphi_v(L) > 1$$

for every $L \in (0,\infty)$. By Proposition 1 and the equivalence of (7) and (4) we conclude that

$$\left(\int_0^L \left(u''^2 + pu'^2 + a\,(x)\,u^2 \right) dx \right)^{1/2}$$

is an equivalent norm in X.

Following the steps of the proof of Theorem 2 in [11], by Lemma 1 it can be proved

Theorem 3. *Let $p > -2\sqrt{a_1}$ where $a_1 := \min_{[0,L]} a\,(x)$. Then the problem (3) possesses infinitely many pairs of solutions $(u_n, -u_n)$ which are critical points of the functional I, and $I\,(u_n) \to +\infty$ as $n \to \infty$.*

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ВЪРХУ СЪЩЕСТВУВАНЕТО НА НЕТРИВИАЛНИ РЕШЕНИЯ НА НЕЛИНЕЙНА ГРАНИЧНА ЗАДАЧА ОТ ЧЕТВЪРТИ РЕД

Юлия Чапарова

Доказан е резултат за съществуване на безбройно много нетривиални решения на нелинейна гранична задача от четвърти ред. Задачата е свързана с намиране на стационарни периодични решения на уравнения на Фишер-Колмогоров и Суифт-Хохенберг, описващи процеси на фазови преходи в особена точка на Лифшиц. Използвани са симетрична теорема за хребета от теорията на критичните точки, както и спектрална теорема за линейните самоспрегнати компактни оператори в хилбертови пространства.