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SPECIAL COMPOSITIONS AND CURVATURE PROPERTIES ON A THREE-DIMENSIONAL WEYL SPACE

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Special compositions, generated by a net in a space with a symmetric linear connection are considered in [2]. In [4] there is introduced the prolonged covariant differentiation of satellites of the metric tensor of a Weyl space. In this paper, the special compositions, generated by a net in the 3-dimensional Weyl space are characterized in terms of the prolonged covariant differentiation. There the form of the curvature tensor on a 3-dimensional Weyl space and the Ricci curvatures of some tangent vectors of the net are given.

1. Preliminaries. Let W_3 be a 3-dimensional Weyl space with a metric tensor g_{ik} and its inverse tensor g^{kj}, i.e. $g_{ik}g^{kj}=\delta_i^j$, $i,j,k=1,2,3$.

As it is well-known [5], the Weyl connection ∇ with components Γ_{ij}^k is determined by the equation:

$$\Gamma_{ij}^k=\left\{ {k \atop ij} \right\}-\left(\omega_i\delta_j^k+\omega_j\delta_i^k-g_{ij}g^{ks}\omega_s\right), \tag{1}$$

where ω_k is the complementary vector of W_3 and $\left\{ {k \atop ij} \right\}$ are the Cristoffel symbols, determined by g_{ij}. There are valid the following equations:

$$\nabla_k g_{ij}=2\omega_k g_{ij}, \qquad \nabla_k g^{ij}=-2\omega_k g^{ij}. \tag{2}$$

Let us consider a composition $W_3(X_2\times X_1)$ in W_3, where X_2 (dim $X_2=2$), X_1 (dim $X_1=1$) are the fundamental manifolds. There exists a unique position of each of the fundamental manifolds X_2 and X_1 at every point $p\in W_3$, which is denoted by $P(X_2)$ and $P(X_1)$, respectively.

According to [8], W_3 is the space of the composition $W_3(X_2\times X_1)$, if there exists a tensor field a_i^j of type (1,1) determined by the equations:

$$a_i^j a_j^k=\delta_i^k, \tag{3}$$

$$N_{ij}^k=a_i^s\nabla_s a_j^k-a_j^s\nabla_s a_i^k+a_s^k\left(\nabla_i a_j^s-\nabla_j a_i^s\right)=0, \tag{4}$$

where N_{ij}^k is the Nijenhuis tensor of the structure a_i^j.

Following [6], the composition $W_3(X_2\times X_1)$ is called Cartesian, if the tangent section of $P(X_2)$ and the tangent vector of $P(X_1)$ is translated parallelly in the direction of every curve of $P(X_2)$ and $P(X_1)$. The characteristics of the Cartesian composition is:

$$\nabla_i a_j^k=0. \tag{5}$$

A composition $W_3\,(X_2 \times X_1)$ is called Chebyshevian, if the tangent section of $P\,(X_2)$ is translated parallelly in $P\,(X_1)$ and the tangent vector of $P\,(X_1)$ is translated parallelly in the direction of every curve of $P\,(X_2)$. The characteristic of the Chebyshevian composition is:

$$\nabla_i a_j^k - \nabla_j a_i^k = 0. \tag{6}$$

The composition $W_3\,(X_2 \times X_1)$ is called geodesic, if the tangent section of $P\,(X_2)$ is translated parallelly in the direction of every curve of $P\,(X_2)$ and the curve $P\,(X_1)$ is geodesic. The characteristic of the geodesic composition is:

$$a_i^k \nabla_s a_k^m + a_s^k \nabla_k a_i^m = 0. \tag{7}$$

Let $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ be a net in W_3, determined by independent tangent vector fields $\underset{k}{\overset{i}{v}}$ of the curve of the net (k =1,2,3). We determine the inverse covectors $\overset{k}{v}_i$ of $\underset{k}{v}^i$ (k =1,2,3), respectively, by the equations:

$$\underset{i}{v}^k \overset{s}{v}_k = \delta_i^s \Leftrightarrow \underset{i}{v}^k \overset{i}{v}_s = \delta_s^k. \tag{8}$$

According to [4], the prolonged covariant differentiation $\overset{\circ}{\nabla}$ of the satellite A with weight $\{p\}$ in the Weyl space is defined by:

$$\underset{i}{\overset{\circ}{\nabla}} A = \nabla_i A - p\omega_i A. \tag{9}$$

In [4] there are found the derivative equations of the directional vectors of the net:

$$\overset{\circ}{\nabla}_i \underset{k}{v}^s = \underset{k}{\overset{m}{T_i}}\, \underset{m}{v}^s, \quad \overset{\circ}{\nabla}_i \overset{k}{v}_s = -\underset{m}{\overset{k}{T_i}}\, \overset{m}{\underset{s}{v}}, \quad k = 1, 2, 3. \tag{10}$$

2. Special compositions in W_3. In [2] there is defined the affinor a_i^k of the composition in the Weyl space. It is determined uniquely by the net and it has the following form in W_3:

$$a_i^k = \underset{1}{v}^k \overset{1}{v}_i + \underset{2}{v}^k \overset{2}{v}_i - \underset{3}{v}^k \overset{3}{v}_i. \tag{11}$$

There is follows immediately a_i^k satisfies (3) and the conditions:

$$a_k^s \underset{1}{v}^k = \underset{1}{v}^s, \quad a_k^s \underset{2}{v}^k = \underset{2}{v}^s, \quad a_k^s \underset{3}{v}^k = -\underset{3}{v}^s. \tag{12}$$

The composition $W_3\,(X_2 \times X_1)$ is determined by a_i^k, if the affinor satisfies (4). The composition $W_3\,(X_2 \times X_1)$ is called associated to the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$.

Theorem 1. *The Weyl space W_3 is a space of the composition $W_3\,(X_2 \times X_1)$ associated to net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ if and only if:*

$$\begin{aligned}
&\underset{3}{\overset{1}{T}}{}_{[k}\overset{3}{v}{}_{i]} + \underset{3}{\overset{1}{T}}{}_s \underset{1}{v}{}^s \overset{3}{v}{}_{[k}\overset{1}{v}{}_{i]} + \underset{3}{\overset{1}{T}}{}_s \underset{2}{v}{}^s \overset{3}{v}{}_{[k}\overset{2}{v}{}_{i]} = 0,\\
&\underset{3}{\overset{2}{T}}{}_{[k}\overset{3}{v}{}_{i]} + \underset{3}{\overset{2}{T}}{}_s \underset{1}{v}{}^s \overset{3}{v}{}_{[k}\overset{1}{v}{}_{i]} + \underset{3}{\overset{2}{T}}{}_s \underset{2}{v}{}^s \overset{3}{v}{}_{[k}\overset{2}{v}{}_{i]} = 0,\\
&\underset{1}{\overset{3}{T}}{}_{[k}\overset{1}{v}{}_{i]} + \underset{2}{\overset{3}{T}}{}_{[k}\overset{2}{v}{}_{i]} - \underset{2}{\overset{3}{T}}{}_s \underset{1}{v}{}^s \overset{2}{v}{}_{[k}\overset{1}{v}{}_{i]} - \underset{1}{\overset{3}{T}}{}_s \underset{2}{v}{}^s \overset{1}{v}{}_{[k}\overset{2}{v}{}_{i]} + \underset{1}{\overset{3}{T}}{}_s \underset{3}{v}{}^s \overset{1}{v}{}_{[k}\overset{3}{v}{}_{i]} + \underset{2}{\overset{3}{T}}{}_s \underset{3}{v}{}^s \overset{2}{v}{}_{[k}\overset{3}{v}{}_{i]} = 0.
\end{aligned} \tag{13}$$

Proof. The affinor a_i^k has weight $\{0\}$ and according to [4] we have:

$$\overset{\circ}{\nabla}_s a_i^k = \nabla_s a_i^k. \tag{14}$$

Then the condition (4) receives the form:

$$a_i^s \overset{\circ}{\nabla}_s a_j^k - a_j^s \overset{\circ}{\nabla}_s a_i^k + a_s^k \left(\overset{\circ}{\nabla}_i a_j^s - \overset{\circ}{\nabla}_j a_i^s \right) = 0. \tag{15}$$

According to (10) and (11) we obtain:

$$\overset{\circ}{\nabla}_s a_j^k = 2 \left(\underset{1}{\overset{3}{T}}_s \underset{3}{v}^k \overset{1}{v}_j - \underset{3}{\overset{1}{T}}_s \underset{1}{v}^k \overset{3}{v}_j + \underset{2}{\overset{3}{T}}_s \underset{3}{v}^k \overset{2}{v}_j - \underset{3}{\overset{2}{T}}_s \underset{2}{v}^k \overset{3}{v}_j \right). \tag{16}$$

Having in mind (11), (12), (16), (15) and the linear independence of $\underset{k}{v}^i$, it follows (13). Conversely, (13) implies (15). □

We receive the following equality by contracting the last equation of (13) with $\underset{1}{v}^s$ and $\underset{2}{v}^s$:

$$\underset{1}{\overset{3}{T}}_s \underset{2}{v}^s = \underset{2}{\overset{3}{T}}_s \underset{1}{v}^s. \tag{17}$$

Theorem 2. *The composition $W_3(X_2 \times X_1)$ is Cartesian if and only if the coefficients of the derivative equations satisfy the conditions:*

$$\underset{1}{\overset{3}{T}}_k = \underset{3}{\overset{1}{T}}_k = \underset{2}{\overset{3}{T}}_k = \underset{3}{\overset{2}{T}}_k = 0. \tag{18}$$

Proof. According (5), (14), (16) and the linear independence of $\underset{k}{v}^i$, $\overset{k}{v}_i$ $(k = 1, 2, 3)$, we obtain (18). Conversely, (18) implies (5).

Since the composition in Theorem 2 is Cartesian, then the vectors $\underset{1}{v}^k$, $\underset{2}{v}^k$ and $\underset{3}{v}^k$ are translated parallelly in the direction of $\underset{3}{v}$ and the vector $\underset{3}{v}^k$ is translated parallelly in the direction of $\underset{1}{v}$ and $\underset{2}{v}$. Hence, according [3], we have the conditions:

$$\underset{1}{v}^k \nabla_k \underset{3}{v}^s = \underset{2}{v}^k \nabla_k \underset{3}{v}^s = \underset{3}{v}^k \nabla_k \underset{3}{v}^s = \underset{3}{v}^k \nabla_k \underset{1}{v}^s = \underset{3}{v}^k \nabla_k \underset{2}{v}^s = 0. \tag{19}$$

Using (9), (10), (18) and (19), we obtain:

Corollary 2.1. *Let the composition $W_3(X_2 \times X_1)$ be Cartesian. There are valid the following conditions for the non-zero coefficients of the derivative equations:*

$$\underset{3}{\overset{3}{T}}_k = 3\omega_k, \quad a_i^k \underset{1}{\overset{2}{T}}_k = \underset{1}{\overset{2}{T}}_i, \quad a_i^k \underset{2}{\overset{1}{T}}_k = \underset{2}{\overset{1}{T}}_i. \tag{20}$$

Theorem 3. *The associated composition $W_3(X_2 \times X_1)$ to the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ is Chebyshevian if and only if:*

$$\begin{gathered} \underset{3}{\overset{1}{T}}_i \overset{3}{v}_k = \underset{3}{\overset{1}{T}}_k \overset{3}{v}_i, \qquad \underset{3}{\overset{2}{T}}_i \overset{3}{v}_k = \underset{3}{\overset{2}{T}}_k \overset{3}{v}_i, \\ \underset{1}{\overset{3}{T}}_i \overset{1}{v}_k - \underset{1}{\overset{3}{T}}_k \overset{1}{v}_i + \underset{2}{\overset{3}{T}}_i \overset{2}{v}_k - \underset{2}{\overset{3}{T}}_i \overset{2}{v}_k = 0. \end{gathered} \tag{21}$$

Proof. According (14), the condition (6) receives the form:

$$\overset{\circ}{\nabla}_i a_k^s - \overset{\circ}{\nabla}_k a_i^s = 0. \tag{22}$$

Having in mind (16), (22) and linear independence of $\underset{k}{v}^i$, we obtain (21). We verify immediately that (21) implies (22), which is equivalent to (6). Hence the composition is Chebyshevian.

It is easy to prove that the coefficients $\overset{1}{\underset{3}{T}}_k$ and $\overset{2}{\underset{3}{T}}_k$ are collinear to $\overset{3}{v}_k$ and $\overset{3}{\underset{1}{T}}_k$, $\overset{3}{\underset{2}{T}}_k$ are linear dependent of $\overset{1}{v}_k$ and $\overset{2}{v}_k$ in the case when $W_3\left(X_2 \times X_1\right)$ is a Chebyshevian composition.

A geodesic composition $W_3\left(X_2 \times X_1\right)$ is characterized by condition (7) and because of (14) it is equivalent to:

$$a_i^k \overset{\circ}{\nabla}_s a_k^m + a_s^k \overset{\circ}{\nabla}_k a_i^m = 0. \tag{23}$$

Having in mind (11), (16) and (23), we establish the truthfulness of the following theorem by an analogous way of the proof of Theorem 3.

Theorem 4. *The associated composition $W_3\left(X_2 \times X_1\right)$ to the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ is geodesic if and only if:*

$$a_i^k \overset{1}{\underset{3}{T}}_k = \overset{1}{\underset{3}{T}}_i, \quad a_i^k \overset{2}{\underset{3}{T}}_k = \overset{2}{\underset{3}{T}}_i, \quad a_i^k \overset{3}{\underset{1}{T}}_k = -\overset{3}{\underset{1}{T}}_i, \quad a_i^k \overset{3}{\underset{2}{T}}_k = -\overset{3}{\underset{2}{T}}_i, \tag{24}$$

i.e. $\overset{1}{\underset{3}{T}}_k$, $\overset{2}{\underset{3}{T}}_k$ belong to $P\left(X_2\right)$ and $\overset{3}{\underset{1}{T}}_k$, $\overset{3}{\underset{2}{T}}_k$ are collinear to covector $\overset{3}{v}_k$.

3. The curvature tensor of W_3. There is known [3], the curvature tensor is expressed by the Ricci tensor and the metric tensor for every 3-dimensional Riemannian manifold. Let $K_{ijk.}{}^{l}$ be the curvature tensor of (W_3, g_{ij}, ω_k) determined by the Cristoffel symbols $\left\{ {k \atop ij} \right\}$. Let K_{jk} and $K = g^{ij} K_{ij}$ be the Ricci tensor and the scalar curvature, respectively. Since the connection $\tilde{\nabla}$ for $\left\{ {k \atop ij} \right\}$ is Riemannian, then the curvature tensor $K_{ijk.}{}^{l}$ has the following form [3, p.282]:

$$K_{ijk.}{}^{s} = g_{jk} K_i^s - g_{ik} K_j^s + K_{jk} \delta_i^s - K_{ik} \delta_j^s - \frac{K}{2}\left(g_{jk}\delta_i^s - g_{ik}\delta_j^s\right), \tag{25}$$

where $K_i^s = g^{sm} K_{im}$.

Theorem 5. *Let (W_3, g, ω) be 3-dimensional Weyl space and ∇ be the Weyl connection of W_3. Then the curvature tensor $R_{ijk.}{}^{s}$ of ∇ satisfies the condition:*

$$R_{ijk.}{}^{s} = \frac{1}{3}\left\{\left(g_{jk} S_{im} - g_{ik} S_{jm}\right) g^{ms} + S_{jk} \delta_i^s - S_{ik} \delta_j^s + \left(S_{ji} - S_{ij}\right) \delta_k^s\right\}, \tag{26}$$

where $S_{jk} = 2R_{jk} + R_{kj} - \dfrac{3R}{4} g_{jk}$, $R = g^{ij} R_{ij}$ – the scalar curvature.

Proof. Having in mind [5] and (1) for $R_{ijk.}{}^{l}$ and tensor $K_{ijk.}{}^{l}$ we have:

$$R_{ijk.}{}^{l} = K_{ijk.}{}^{l} - \tilde{\nabla}_i T_{jk}^{\ l} + \tilde{\nabla}_j T_{ik}^{\ l} + T_{is}^{\ l} T_{jk}^{\ s} - T_{js}^{\ l} T_{ik}^{\ s},$$

where $T_{jk}^{\ l} = \omega_j \delta_k^{?l} + \omega_i \delta_j^{?l} - g_{jk} g^{lm} \omega_m$.

Using (2), the last equation has the following form:

$$R_{ijk.}{}^{l} = K_{ijk.}{}^{l} + \left(\tilde{\nabla}_j \omega_i - \tilde{\nabla}_i \omega_j\right) \delta_k^l + M_{jk} \delta_i^l - M_{ik} \delta_j^l + g_{jk} M_{i.}^l - g_{ik} M_{j.}^l, \tag{27}$$

where $M_{jk} = \tilde{\nabla}_j \omega_k + \omega_j \omega_k - \dfrac{1}{2} g_{jk} \omega^s \omega_s$, $M_{j.}^{\ l} = g^{lk} M_{jk}$.

According to (1) and the identity for the curvature tensor of a Weyl space [5], i.e. $R^i_{ski} = -2n\nabla_{[s\omega_k]}$, we have the following equality for $n = 3$

$$\nabla_j \omega_i - \nabla_i \omega_j = \tilde{\nabla}_j \omega_i - \tilde{\nabla}_i \omega_j = \frac{R_{ji} - R_{ij}}{3}. \tag{28}$$

Then, by contracting i and l in (27), we obtain the form of the Ricci tensor and the scalar curvature:

$$K_{jk} = \frac{2}{3} R_{jk} + \frac{1}{3} R_{kj} - M_{jk} - g_{jk} g^{is} M_{is}, \quad R = K + 4 g^{is} M_{is}. \tag{29}$$

The equalities (28), (25), (29) and (27) imply (26). □

There is known [7], the Ricci curvature in direction the unit vector field v^k is $R(v, v) = R_{ij} v^i v^j$ with respect to g_{ij}.

Theorem 6. *Let $W_3 (X_2 \times X_1)$ be a Cartesian composition. Then the Ricci curvature in the direction of the vector $\underset{3}{v}^k$ is zero.*

Proof. The condition (20) for Cartesian composition implies $\overset{\circ}{\nabla}_j \underset{3}{v}^s = 3\omega_j \underset{3}{v}^s$. Since the vector $\underset{3}{v}^s$ has weight $\{-1\}$, according to (9) and the last equation, we obtain $\nabla_j \underset{3}{v}^s = 2\omega_j \underset{3}{v}^s$. The integrability conditions of the last equation are:

$$\nabla_i \nabla_j \underset{3}{v}^s - \nabla_j \nabla_i \underset{3}{v}^s = R_{ijk.}^{\ \ \ s} \underset{3}{v}^k. \tag{30}$$

We obtains the left side of (30), by using of (28), and the right side of (30) – by using of (26). After some calculation, we obtain $R_{jk} \underset{3}{v}^j \underset{3}{v}^k = 0$, i.e. the Ricci curvature in the direction of $\underset{3}{v}^k$ is zero.

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СПЕЦИАЛНИ КОМПОЗИЦИИ В ТРИМЕРНО ВАЙЛОВО ПРОСТРАНСТВО. СВОЙСТВА НА ТЕНЗОРА НА КРИВИНА

Добринка К. Грибачева, Георги Зл. Златанов

Специални композиции, породени от мрежа в пространство със симетрична линейна свързаност се изучават в [2]. В [4] се въвежда продължено ковариантно диференциране на спътниците на метричния тензор на Вайлово пространство. В тази работа с помощта на продълженото ковариантно диференциране се характеризират специални композиции, породени от мрежа в тримерно Вайлово пространство. Намерен е вида на тензора на кривина на тримерно Вайлово пространство и кривините на Ричи на някои от допирателните вектори на линиите на мрежата.