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TWO-POINT BOUNDARY-VALUE PROBLEMS WITH IMPULSE EFFECTS*

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A new scheme for investigation of boundary value problems of ordinary differential equations with impulse effects at finite number of points is suggested. The problem is reduced to a two-point boundary value problem which dimension is greater than the dimension of the original problem. Conditions for existence of unique solution and family of solutions are obtained.

1. Statement of the problem. We consider a two-point boundary-value problem with generalized impulse conditions at finite number of points

$$\frac{dy}{dx} = A(x)y + f(x),\ x \in [a, b],\ x \neq \tau_i, \tag{1}$$

$$By(a) + Cy(b) = d, \tag{2}$$

$$N_i y(\tau_i + 0) + M_i y(\tau_i - 0) = v_i,\ i = \overline{1, p}, \tag{3}$$

$$a = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_p < \tau_{p+1} = b,$$

where the coefficients of the system (1) and the equalities (2), (3) are subordinate to the following conditions:

(H1) $A(x)$ is $(n \times n)$ matrix with continuous elements on $[a, b]$, $f(x)$ is n-dimensional partially continuous vector function with break points of the first kind at τ_i: $f(x) = f_1(x),\ x \in [a, \tau_1],\ f(x) = f_i(x),\ x \in (\tau_i, \tau_{i+1}],\ i = \overline{1, p}$, where $f_i(x)$ is continuous in $[\tau_{i-1}, \tau_i]$, $i = \overline{1, p+1}$;

(H2) B and C are $(k \times n)$ constant matrices, $d \in \mathbf{R}^k$ and M_i, N_i $i = \overline{1, p}$, are $(s \times n)$ constant matrices, $v_i \in \mathbf{R}^s$.

Further, we give conditions for existence of the solution $y(\cdot) \in C^1([a, b] \setminus \{\tau_1, \cdots, \tau_p\})$ of the impulsive boundary-value problem (1) – (3) and construct this solution.

If instead of (3) we consider the following impulse conditions

$$y(\tau_i + 0) + (E_n + S_i)\, y(\tau_i - 0) = v_i,\ i = \overline{1, p},$$

where $(E_n + S_i)$ are nonsingular matrices, then we obtain the problem investigated in [4]. The system (1) with impulse and boundary conditions united in the form $\sum\limits_{i=1}^{p+1} l_i y_i(\cdot)$

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where the functionals

$$l_i y_i(\cdot) = \int_{\tau_{i1}}^{\tau_i} [d\sigma(s)]\, C_i(s) y_i(s),\ i = \overline{1, p+1},$$

is investigated in [2].

2. Main results. We denote $x_i = \tau_i$, $i = \overline{0, p+1}$ and replace

$$\frac{x - x_{i-1}}{h_i} = t,\ h_i = x_i - x_{i-1},\ i = \overline{1, p+1}. \tag{4}$$

This means that $x \in [x_{i-1}, x_i]$ is equivalent to $t \in [0,1]$. Therefore after change of variables (4) the intervals $[x_{i-1}, x_i]$, $i = \overline{1, p+1}$ are represented in $[0,1]$.

Let $y(x_{i-1} + th_i) = z_i(t)$, $t \in [0,1]$ when $x \in [x_{i-1}, x_i]$. Then $\frac{dy}{dx} = \frac{dz_i(t)}{dt} \frac{dt}{dx} = \frac{1}{h_i} \frac{dz_i(t)}{dt}$. Thus (1) takes the form

$$\frac{dz_i(t)}{dt} = h_i A(x_{i-1} + th_i) z_i(t) + h_i f(x_{i-1} + th_i),\ i = \overline{1, p+1}.$$

We introduce the next denotations

$$A_i(t) = h_i A(x_{i-1} + th_i), \quad g_i(t) = h_i f(x_{i-1} + th_i).$$

Then the last system takes the form

$$\frac{dz_i(t)}{dt} = A_i(t) z_i(t) + g_i(t),\ i = \overline{1, p+1},\ t \in [0,1]. \tag{5}$$

By means of the notations introduced above we find $y(a) = z_1(0)$, $y(b) = z_{p+1}(1)$. For this reason (2) takes the form

$$B z_1(0) + C z_{p+1}(1) = d. \tag{6}$$

Since the solution $y(x)$ of the problem (1) – (3) is continuous on every interval $[a, x_1]$, $(x_{i-1}, x_i]$, $i = \overline{2, p+1}$, then as $\lim\limits_{x \to \tau_i + 0} y(x) = y(\tau_i + 0)$, we obtain

$$y(\tau_i - 0) = z_i(1), \quad y(\tau_i + 0) = z_{i+1}(0).$$

On this way the impulse conditions (3) take the form

$$M_i z_i(1) + N_i z_{i+1}(0) = v_i,\ i = \overline{1, p}. \tag{7}$$

Let $z(t)$ and $g(t)$ be $(p+1)n$-dimensional vector functions

$$z(t) = col(z_1(t), \cdots, z_{p+1}(t)), \quad g(t) = col(g_1(t), \cdots, g_{p+1}(t)).$$

Then we rewrite the differential systems (5) and conditions (6), (7) like a generalized two-point boundary-value problem

$$\dot{z}(t) = \overline{A}(t) z(t) + g(t), \quad t \in [0,1], \tag{8}$$

$$\overline{B} z(0) + \overline{C} z(1) = \overline{d}, \tag{9}$$

where

$$\overline{A}(t) = \operatorname{diag}(A_1(t), \cdots, A_{p+1}(t)) \text{ is } ((p+1)n \times (p+1)n) \text{ matrix},$$

$$\overline{B} = \operatorname{diag}(B, N_1, \cdots, N_p)) \text{ is } ((k+ps) \times (p+1)n)) \text{ constant matrix},$$

$$\overline{C}=\begin{pmatrix} 0 & \cdots & \cdots & \cdots & C \\ M_1 & 0 & & \text{\Large 0} & \\ 0 & M_2 & 0 & & \\ & \text{\Large 0} & \ddots & \ddots & \\ & & & M_p & 0 \end{pmatrix} \text{ is } ((k+ps)\times(p+1)n) \text{ constant matrix,}$$

$$\overline{d}=[d,\ v_1,\ \cdots,v_p]^T \text{ is } (k+ps)\text{- dimensional vector.}$$

We are to solve the problem (8), (9) instead of the problem (1) – (3). It is clear that $y_i(x)=[z(t)]_{n_i}$ when $t=(x-\tau_i)/h_i$. Here index n_i means successively n in number components of the solution $z(t)$ of (8), (9).

Let $\overline{\Phi}(t),\ \overline{\Phi}(0)=E_{(p+1)n}$ be the normal fundamental matrix of the solutions of $\dot{z}=\overline{A}(t)z$. The generalized solution of the system (8) has the form

(10) $$z(t)=\overline{\Phi}(t)c+\eta(t),\ \ c\in\mathbf{R}^{(p+1)n},$$

where $\eta(t)=\int\limits_0^t\overline{\Phi}(t)\overline{\Phi}^{-1}(s)g(s)ds$ is a particular solution of (8). We substitute (10) in the boundary condition (9) and bearing in mind $\overline{\Phi}(0)=E_{(p+1)n}$, $\eta(0)=0$, we obtain

(11) $$Qc=\overline{d}-\overline{C}\eta(1),$$

where $Q=\overline{B}+\overline{C}\overline{\Phi}(1)$ is $((k+ps)\times(p+1)n)$ matrix.

We denote by Q^+ the $((p+1)n\times(k+ps))$ pseudoinverse matrix [1], [3] of the matrix Q, by P_Q and P_{Q^*} the orthoprojectors $P_Q:\mathbf{R}^{(p+1)n}\to\ker(Q)$, $P_{Q^*}:\mathbf{R}^{k+ps}\to\ker(Q^*)$, $Q^*=Q^T$.

Theorem. *Let the conditions* (H1), (H2) *be satisfied and* rank$Q=n_1<\min(k+ps,(p+1)n)$. *Then the boundary-value problem* (8), (9) *has one-parametric family of solutions*

(12) $$z(t,\xi)=\overline{\Phi}(t)P_Q\xi+\overline{z}(t),\ \ \overline{z}(t)=\overline{\Phi}(t)Q^+(\overline{d}-\overline{C}\eta(1))+\eta(t)$$

if and only if $P_{Q^*}\left(\overline{d}-\overline{C}\eta(1)\right)=0$.

Proof. We have rank$Q=n_1<\min(k+ps,(p+1)n)$. Then the system (11) has a parametric solution

(13) $$c=P_Q\xi+Q^+\left(\overline{d}-\overline{C}\eta(1)\right),\ \ \xi\in\mathbf{R}^{(p+1)n}$$

if and only if the condition of orthogonality $P_{Q^*}\left(\overline{d}-\overline{C}\eta(1)\right)=0$ is fulfilled. We substitute (13) in (10) and obtain (12).

Corollary 1. *Let the conditions* (H1), (H2) *be satisfied and* $P_{Q^*}=0$. *Then the boundary-value problem* (8), (9) *has one-parametric solution of the form* (12).

In this case rank$Q=(p+1)n$ and the system (11) is always solvable.

Corollary 2. *Let the conditions* (H1), (H2) *be satisfied and* $P_Q=0$. *Then the boundary-value problem* (8), (9) *has an unique solution* $z(t)=\overline{z}(t)$ *if and only if*

$P_{Q^*}\left(\overline{d}-\overline{C}\eta(1)\right)=0.$

In this case $\operatorname{rank}Q=k+ps$.

Corollary 3. *Let the conditions* (H1), (H2) *be satisfied and* $P_Q=0$, $P_{Q^*}=0$. *Then the boundary-value problem* (8), (9) *has a unique solution* $z(t)=\overline{z}(t)$ *and* $Q^+=Q^{-1}$.

In this case $k+ps=(p+1)n$ and $\det Q\neq 0$. Here we may supplement a case when $k=s=n$.

Keeping in mind the replacement (4) for the solution of the boundary-value problem (1), (2) with impulse effects (3), we find

$$y(x,\xi)=\begin{cases} y_1(x)=\left[\overline{\Phi}\left(\dfrac{x-a}{h_1}\right)P_Q\right]_{n_1}\xi+\left[\overline{z}\left(\dfrac{x-a}{h_1}\right)\right]_{n_1}, & x\in[a,\tau_1],\\ y_i(x)=\left[\overline{\Phi}\left(\dfrac{x-\tau_i}{h_i}\right)P_Q\right]_{n_i}\xi+\left[\overline{z}\left(\dfrac{x-\tau_i}{h_i}\right)\right]_{n_i}, & x\in(\tau_{i-1},\tau_i],\\ & i=\overline{2,p+1}, \end{cases} \tag{14}$$

where $n_1+n_2+\cdots+n_{p+1}=(p+1)n$, n_1 means the first n in number rows, n_2- the second and etc., n_{p+1}- the last n in number rows of the matrix $\overline{\Phi}(t)P_Q$ and the vector $\overline{z}(t)$.

3. Examples.

3.1. We illustrate the theorem by the impulsive problem

$$\frac{dy}{dx}=0,\ x\in[0,2],\ x\neq 1,\ y(0)+y(2)=1,\ y(1+0)+y(1-0)=1.$$

In this case

$$\dot z(t)=\begin{bmatrix}\dot z_1(t)\\ \dot z_2(t)\end{bmatrix},\ g(t)=\begin{bmatrix}0\\0\end{bmatrix},\ \overline{\Phi}(t)=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \overline{B}=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

$$\overline{C}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \overline{d}=\begin{bmatrix}1\\1\end{bmatrix},\ \eta(t)=\begin{bmatrix}0\\0\end{bmatrix}.$$

Since Q has the representation $Q=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, then $\operatorname{rank}Q=1$. Thus we find successively

$$Q^+=\frac{1}{4}\begin{bmatrix}1&1\\1&1\end{bmatrix},\ P_Q=\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix},\ P_{Q^*}=\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}.$$

Obviously, the condition of orthogonality $P_{Q^*}\begin{bmatrix}1\\1\end{bmatrix}=0$ is fulfilled. This shows that the system

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}c=\begin{bmatrix}1\\1\end{bmatrix}$$

is always solvable and according to (13) has one parametric solution

$$c=\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\xi+\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix},\ \xi\in\mathbf{R}^2.$$

From (12) we find

$$z(t,\xi)=\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\xi+\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix},\ \xi\in\mathbf{R}^2.$$

Finally from (14) we get

$$x(t,\xi) = \begin{cases} y_1(x) = \dfrac{1}{2}\left[\ \ 1, \ \ -1\right]\xi + \dfrac{1}{2}, & x \in [0,1], \\ \\ y_2(x) = \dfrac{1}{2}\left[-1, \ \ \ \ 1\right]\xi + 12, & x \in (1,2] \end{cases}$$

$$x(t) = \begin{cases} y_1(x) = \ \ \lambda + 12, & x \in [0,1], \\ \\ y_2(x) = -\lambda + 12, & x \in (1,2]. \end{cases} \quad \lambda \in \mathbf{R}$$

3.2. We will find periodic solution of the system

$$\frac{dy}{dx} = 1,\ x \in [0,3],\ x \neq 1,\ y(0) = y(3),\ y(1+0) = 0.$$

In this case $h_1 = 1,\ h_2 = 2$ and the system (8) has the form

$$\dot{z}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and we obtain $\overline{\Phi}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The boundary conditions (9) are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z(0) + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} z(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The system (11) has the form

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} c = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Then

$$c = \begin{bmatrix} 2, & 0 \end{bmatrix}^T.$$

In accordance with corollary 3

$$z(t) = \overline{z}(t) = \begin{bmatrix} t+2, & 2t \end{bmatrix}^T$$

and from (14) we find

$$x(t,\xi) = \begin{cases} y_1(x) = x + 2, & x \in [0,1], \\ \\ y_2(x) = x - 1, & x \in (1,3]. \end{cases}$$

We describe a scheme for investigation of boundary-value problems with impulse effects of the form (3). This scheme may be applied to multipoint boundary-value problems. Then the matrices $\overline{B}$ and $\overline{C}$ are not so simple.

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ДВУТОЧКОВИ ГРАНИЧНИ ЗАДАЧИ С ИМПУЛСНО ВЪЗДЕЙСТИВИЕ

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В работата е предложена нова схема за изследване на гранични задачи за обикновени диференциални уравнения с общи импулсни въздействия в краен брой точки чрез свеждане по двуточкова гранична задача с по-голяма размерност от първоначалната. Получени са условията за съществуване както на единствено решение, така и на семейство от решения.