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CURVATURE TENSORS ON SOME FIVE–DIMENSIONAL ALMOST CONTACT B-METRIC MANIFOLDS*

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There are considered 5-dimensional almost contact B-metric manifolds of two basic classes. It is proved that every manifold from the section of these classes is with point-wise constant sectional curvatures. It is studied the curvature tensor of the manifolds of these two classes and some their curvature characteristics are given.

1. Preliminaries. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact manifold with B-metric, i.e. (φ, ξ, η) is an almost contact structure and g is a metric on M such that:

$$\varphi^2 = -id + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad g(\varphi\cdot, \varphi\cdot) = -g(\cdot,\cdot) + \eta(\cdot)\eta(\cdot).$$

Both metrics g and its associated $\tilde{g} : \tilde{g}(\cdot,\cdot) = g(\cdot, \varphi\cdot) + \eta(\cdot)\eta(\cdot)$ are indefinite metrics of signature $(n+1, n)$ [1].

Further, X, Y, Z, W will stand for arbitrary differentiable vector fields on M (i.e. X, Y, Z, $W \in \mathfrak{X}(M)$), and x, y, z, w – arbitrary vectors in the tangential space T_pM to M at some point $p \in M$.

Let $(V, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional vector space with almost contact B-metric structure. Let us denote the subspace $hV := \ker \eta$ of V, and the restrictions of g and φ on hV by the same letters. It is obtained a $2n$-dimensional vector space hV with a complex structure φ and B-metric g. Let $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi\}$ be an adapted φ-basis of V, where $-g(e_i, e_j) = g(\varphi e_i, \varphi e_j) = \delta_{ij}$, $g(e_i, \varphi e_j) = 0$, $\eta(e_i) = 0$; $i, j \in \{1, \dots, n\}$.

A decomposition of the class of the almost contact manifolds with B-metric with respect to the tensor $F : F(X, Y, Z) = g((\nabla_X \varphi)Y, Z)$ is given in [1], where there are defined eleven basic classes $\mathcal{F}_i$ $(i = 1, \dots, 11)$. The Levi-Civita connection of g is denoted by ∇. The special class $\mathcal{F}_0$: $F = 0$ is contained in each $\mathcal{F}_i$. The following 1-forms are associated with F:

$$\theta(\cdot) = g^{ij} F(e_i, e_j, \cdot), \quad \theta^*(\cdot) = g^{ij} F(e_i, \varphi e_j, \cdot), \quad \omega(\cdot) = F(\xi, \xi, \cdot),$$

where $\{e_i, \xi\}$ $(i = 1, \dots, 2n)$ is a basis of T_pM, and (g^{ij}) is the inverse matrix of (g_{ij}).

In this paper we consider especially the classes $\mathcal{F}_4$ and $\mathcal{F}_5$ arise from the main components of F. There are known explicit examples of $\mathcal{F}_5$- and $(\mathcal{F}_4 \oplus \mathcal{F}_5)$-manifolds in

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[1]. Moreover, these classes are analogues to the classes of the known α-Sasakian and $\tilde{\alpha}$-Kenmotsu manifolds in the geometry of the almost contact metric manifolds. The considered classes are determined by the conditions

$$\begin{aligned} &\mathcal{F}_4 : F(X,Y,Z) = -\tfrac{\theta(\xi)}{2n}\left\{g(\varphi X,\varphi Y)\eta(Z) + g(\varphi X,\varphi Z)\eta(Y)\right\}, \\ &\mathcal{F}_5 : F(X,Y,Z) = -\tfrac{\theta^*(\xi)}{2n}\left\{g(X,\varphi Y)\eta(Z) + g(X,\varphi Z)\eta(Y)\right\}. \end{aligned} \tag{1.1}$$

The structural 1-form η is closed on the $\mathcal{F}_i$-manifolds ($i = 4, 5$).

An important problem in the differential geometry of such manifolds is the studying of the manifolds with constant totally real sectional curvatures. In this paper we pay attention to the $\mathcal{F}_i$-manifolds ($i = 4, 5$) of dimension 5. This is the boundary dimension for the necessary and sufficient condition $\mathcal{F}_0$-manifold to be with point-wise constant sectional curvatures [7].

The following transformation is called a contact-conformal transformation

$$c : \bar{g}(X,Y) = e^{2u}\cos 2v g(X,Y) + e^{2u}\sin 2v g(X,\varphi Y) + (1 - e^{2u}\cos 2v)\eta(X)\eta(Y), \tag{1.2}$$

where u and v are differentiable functions on M. These transformations form a group denoted by C. The manifolds (M,φ,ξ,η,g) and $(\bar{M},\varphi,\xi,\eta,\bar{g})$ are called C-equivalent manifolds [3].

As it is known [3], the subclass $\mathcal{F}_i^0 \subset \mathcal{F}_i$ is the class of the C_i^0-equivalent manifolds to $\mathcal{F}_0$ ($i = 4, 5$). These subclasses of $\mathcal{F}_i$ and these subgroups of C are determined by the conditions:

$$\begin{aligned} &\mathcal{F}_4^0 = \{\mathcal{F}_4 \mid \mathrm{d}\theta = 0\}, \qquad C_4^0 = \{c \in C \mid \mathrm{d}u = \mathrm{d}v \circ \varphi,\ \mathrm{d}(\mathrm{d}v(\xi)) = 0\}, \\ &\mathcal{F}_5^0 = \{\mathcal{F}_5 \mid \mathrm{d}\theta^* = 0\}, \qquad C_5^0 = \{c \in C \mid \mathrm{d}v = -\mathrm{d}u \circ \varphi,\ \mathrm{d}(\mathrm{d}u(\xi)) = 0\}. \end{aligned} \tag{1.3}$$

The corresponding 1-forms of $\bar{F}$ on $\bar{M}$ are

$$\bar{\theta} = 2n\mathrm{d}v(\xi)\eta \quad \text{for } i = 4; \qquad \bar{\theta}^* = 2n\mathrm{d}u(\xi)\eta \quad \text{for } i = 5. \tag{1.4}$$

The relations between the corresponding Levi-Civita connections are [4]:

$$\begin{aligned} \bar{\nabla}_X Y = \nabla_X Y &- \mathrm{d}v(\varphi X)\varphi^2 Y - \mathrm{d}v(\varphi Y)\varphi^2 X + \mathrm{d}v(X)\varphi Y + \mathrm{d}v(Y)\varphi X \\ &+ \left[\varphi\mathrm{grad}(v) - e^{2u}\sin 2v\mathrm{d}v(\xi)\xi\right] g(\varphi X,\varphi Y) \\ &- \left[\mathrm{grad}(v) - (1 - e^{2u}\cos 2v)\mathrm{d}v(\xi)\xi\right] g(X,\varphi Y) \qquad \text{for } i = 4; \end{aligned} \tag{1.5}$$

$$\begin{aligned} \bar{\nabla}_X Y = \nabla_X Y &- \mathrm{d}u(X)\varphi^2 Y - \mathrm{d}u(Y)\varphi^2 X - \mathrm{d}u(\varphi X)\varphi Y - \mathrm{d}u(\varphi Y)\varphi X \\ &+ \left[\mathrm{grad}(u) - (1 - e^{2u}\cos 2v)\mathrm{d}u(\xi)\xi\right] g(\varphi X,\varphi Y) \\ &+ \left[\varphi\mathrm{grad}(u) - e^{2u}\sin 2v\mathrm{d}u(\xi)\xi\right] g(X,\varphi Y) \qquad \text{for } i = 5. \end{aligned} \tag{1.6}$$

2. Curvature tensors. Let R and $\bar{R}$ be the C-corresponding curvature tensors for ∇ and $\bar{\nabla}$, respectively.

Let $\mathcal{R}$ be the set of all curvature-like tensors with the properties of R:

$$R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z), \quad \underset{x,y,z}{\mathfrak{S}} R(x,y)z = 0, \tag{2.1}$$

$$\underset{x,y,z}{\mathfrak{S}} (\nabla_x R)(y,z)w = 0. \tag{2.2}$$

The corresponding Ricci tensor and scalar curvatures are denoted respectively by:

$$\rho(y,z) = g^{ij}R(e_i,y,z,e_j), \qquad \tau = g^{ij}\rho(e_i,e_j), \qquad \tau^* = \varphi_k^j g^{ik}\rho(e_i,e_j),$$

where $\{e_i\}_{i=1}^{2n+1}$ is a basis of T_pM.

We use the following curvature-like tensors, which are invariant with respect to the structural group. The tensor S is a symmetric and φ-antiinvariant tensor of type $(0,2)$.

$$\psi_1(S)(x,y,z,u) = g(y,z)S(x,u) - g(x,z)S(y,u) + g(x,u)S(y,z) - g(y,u)S(x,z),$$

$$\psi_2(S)(x,y,z,u) = \psi_1(S)(x,y,\varphi z,\varphi u),$$

$$\psi_3(S)(x,y,z,u) = -\psi_1(S)(x,y,\varphi z,u) - \psi_1(S)(x,y,z,\varphi u),$$

$$\psi_4(S)(x,y,z,u) = \psi_1(S)(x,y,\xi,u)\eta(z) + \psi_1(S)(x,y,z,\xi)\eta(u),$$

$$\psi_5(S)(x,y,z,u) = \psi_1(S)(x,y,\xi,\varphi u)\eta(z) + \psi_1(S)(x,y,\varphi z,\xi)\eta(u).$$

We denote tensors $\pi_i = \frac{1}{2}\psi_i(g)$ $(i = 1,2,3), \pi_i = \psi_i(g)$ $(i = 4,5)$. The tensors $\bar{\psi}_i(S)$ and $\bar{\pi}_i$ are the corresponding tensors with respect to $\bar{g}$ $(i = 1,\dots,5)$.

A decomposition of $\mathcal{R}$ over (V,φ,ξ,η,g) into 20 mutually orthogonal and invariant factors with respect to the structural group $GL(n,C)\cap O(n,n))\times I$ is obtained in [5]. It is received initially the partial decomposition $\mathcal{R} = h\mathcal{R}\oplus v\mathcal{R}\oplus w\mathcal{R}$ and subsequently the decompositions:

$$h\mathcal{R} = \omega_1\oplus\ldots\oplus\omega_{11}, \quad v\mathcal{R} = v_1\oplus\ldots\oplus v_5, \quad w\mathcal{R} = w_1\oplus\ldots\oplus w_4.$$

The characteristic conditions of the factors ω_i $(i = 1,\dots,11)$, v_j $(j = 1,\dots,5)$, w_k $(k = 1,\dots,4)$ are given in [5]. Let us recall [6], an almost contact B-metric manifold is said to be in one of the classes $h\mathcal{R}_i$, $h\mathcal{R}_i^\perp$, $v\mathcal{R}_j$, $v\mathcal{R}_j^\perp$, $w\mathcal{R}$, ω_k, v_r, w_s if R belongs to the corresponding component, where $i = 1,2,3$; $j = 1,2$; $k = 1,\dots,11$; $r = 1,\dots,5$; $s = 1,\dots,4$.

From the decomposition of $\mathcal{R}$ follows that the 5-dimensional almost contact B-metric manifold cannot belong to the factors ω_3 and ω_4.

Let (M,φ,ξ,η,g) be a 5-dimensional manifold. Moreover, let $k(\alpha;p)$, $\tilde{k}(\alpha;p)$ be the scalar curvatures of a nondegenerate totally real orthogonal to ξ section α (i.e. $\alpha\perp\varphi\alpha$, $\alpha\perp\xi$) in T_pM, $p\in M$. In this connection let us recall the following

Theorem 2.1 ([7]). *Let (M,φ,ξ,η,g) $(\dim M \geq 5)$ be an $\mathcal{F}_0$-manifold. M is of constant totally real sectional curvatures $\nu(p) = k(\alpha;p)$ and $\tilde{\nu}(p) = \tilde{k}(\alpha;p)$ if and only if*

$$R = \nu\left[\pi_1 - \pi_2 - \pi_4\right] + \tilde{\nu}\left[\pi_3 + \pi_5\right].$$

Both functions ν and $\tilde{\nu}$ are constant if M is connected and $\dim M \geq 7$.

We have in mind that R satisfies the Kähler property on every $\mathcal{F}_0$-manifold

$$R(X,Y,\varphi Z,\varphi W) = -R(X,Y,Z,W). \tag{2.3}$$

According to the decomposition of $\mathcal{R}$ in [5] we obtain the equivalence of the R's expression in the last theorem and the condition $R \in \omega_1 \oplus \omega_2$ for dimension 5. Then we have

Theorem 2.2 *Every 5-dimensional $\mathcal{F}_0$-manifold has point-wise constant sectional curvatures $\nu(p) = k(\alpha;p)$, $\tilde\nu(p) = \tilde k(\alpha;p)$ and it belongs to $\omega_1 \oplus \omega_2$.*

Proof. Let $\{e_1, e_2, \varphi e_1, \varphi e_2, \xi\}$ be a φ-basis of T_pM. Then $x = x^i e_i + x^{i+2}\varphi e_i + \eta(x)\xi$, $(i = 1,2)$. Using the properties (2.1) and (2.3) of $R(x,y,z,w)$ we compute immediately

$$R = \nu\,[\pi_1 - \pi_2 - \pi_4] + \tilde\nu\,[\pi_3 + \pi_5]\,, \quad \nu = R(e_1,e_2,e_2,e_1),\ \tilde\nu = R(e_1,e_2,e_2,\varphi e_1).$$

Then according Theorem 2.1. we establish the point-wise constancy for α.

Immediately it follows that $R \in \omega_1 \oplus \omega_2$ and consequently $M \in \omega_1 \oplus \omega_2$. □

Lemma 2.3. *Let $(\bar M, \varphi, \xi, \eta, \bar g)$ be a 5-dimensional C_i^0-equivalent $\mathcal{F}_i^0$-manifold to an $\mathcal{F}_0$-manifold with curvatures $\nu = \nu(p)$, $\tilde\nu = \tilde\nu(p)$ of α $(i = 4,5)$. The curvature tensor on $\bar M$ has the following form*

$$\begin{aligned}\bar R = &-e^{-2u}\cos 2v\,\left[\bar\psi_1 - \bar\psi_2 - \bar\psi_4\right](S) - e^{-2u}\sin 2v\,\left[\bar\psi_3 + \bar\psi_5\right](S) - \bar A\\ &+ e^{-4u}\left\{\nu\cos 4v - \tilde\nu\sin 4v\right\}\left[\bar\pi_1 - \bar\pi_2 - \bar\pi_4\right] + e^{-4u}\left\{\nu\sin 4v + \tilde\nu\cos 4v\right\}\left[\bar\pi_3 + \bar\pi_5\right],\end{aligned}$$

where

$$S(Y,Z) = (\nabla_Y\sigma)\,Z + \sigma(\varphi Y)\sigma(\varphi Z) - \sigma(Y)\sigma(Z) - \tfrac12\sigma(s)g(\varphi Y,\varphi Z) - \tfrac12\sigma(\varphi s)g(Y,\varphi Z),$$

$$\operatorname{tr} S = g^{ij}S_{ij} = \Delta u, \quad \operatorname{tr}^* S = \varphi_k^j g^{ik} S_{ij} = -\Delta v, \quad \Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} + \frac{\partial^2}{\partial x_5^2}$$

is the Laplacian and we have for $\bar A$ and $\sigma = g(s,\cdot)$, respectively,

a) for $i = 4$: $\quad \bar A = (\mathrm{d}v(\xi))^2\left[\bar\pi_2 - \bar\pi_4\right], \qquad \sigma = \mathrm{d}v\circ\varphi;$

b) for $i = 5$: $\quad \bar A = (\mathrm{d}u(\xi))^2\,\bar\pi_1, \qquad \sigma = -\mathrm{d}u\circ\varphi^2.$

Then we obtain the corresponding Ricci tensor and scalar curvatures, respectively:

$$\begin{aligned}\bar\rho = &\left[-e^{-2u}(\Delta u\cos 2v + \Delta v\sin 2v) + 2e^{-4u}(\nu\cos 4v - \tilde\nu\sin 4v)\right]\bar g\\ &+ \left[e^{-2u}(\Delta u\sin 2v - \Delta v\cos 2v) - 2e^{-4u}(\nu\sin 4v + \tilde\nu\cos 4v)\right]\bar{\tilde g}\\ &+ \left[-e^{-2u}\big(\Delta u(\cos 2v - \sin 2v) + \Delta v(\cos 2v + \sin 2v)\big)\right.\\ &\quad\left.-2e^{-4u}\big((\nu - \tilde\nu)\cos 4v - (\nu + \tilde\nu)\sin 4v\big)\right]\eta\otimes\eta - \bar\rho(\bar A),\\ \bar\tau = &-4\left[e^{-2u}(\Delta u\cos 2v + \Delta v\sin 2v) - 2e^{-4u}(\nu\cos 4v - \tilde\nu\sin 4v)\right] - \bar\tau(\bar A),\\ \bar\tau^* = &-4\left[e^{-2u}(\Delta u\sin 2v - \Delta v\cos 2v) - 2e^{-4u}(\nu\sin 4v + \tilde\nu\cos 4v)\right].\end{aligned} \tag{2.4}$$

where $\quad \bar\rho(\bar A) = -4(\mathrm{d}v(\xi))^2\eta\otimes\eta, \quad \bar\tau(\bar A) = -4(\mathrm{d}v(\xi))^2 \quad$ for $i = 4$;

$\qquad \bar\rho(\bar A) = 4(\mathrm{d}u(\xi))^2\bar g, \qquad \bar\tau(\bar A) = 20(\mathrm{d}u(\xi))^2 \quad$ for $i = 5$.

The obtained $\bar R$ as a curvature tensor has to satisfy the second Bianchi identity (2.2). As a consequence it is known the following corollary of the mentioned identity in local coordinates

$$\nabla_i \tau = 2\nabla_j \rho_i^j. \tag{2.5}$$

Applying (2.5) for the tensors from (2.4) in the case for $i = 4$ we get $\mathrm{d}v(\xi) = 0$. This equality implies the following conclusion.

Theorem 2.4 *The 5-dimensional manifold* $(\bar{M}, \varphi, \xi, \eta, \bar{g}) = C_4^0(M, \varphi, \xi, \eta, g)$ *is an* $\mathcal{F}_0$*-manifold, i.e. it is not possible to be obtained nontrivial 5-dimensional* $\mathcal{F}_4^0$*-manifold by* C*-transformation.*

In the same way (when $i = 5$), we get the condition $v = \frac{1}{4}\arctan(\nu/\tilde{\nu})$ and consequently $\Delta u = \Delta v = 0$ for the functions determining the C_5^0-transformation and then we receive

Lemma 2.5 *There are valid the following equalities for a 5-dimensional* $\mathcal{F}_5^0$*-manifold which is* C_4^0*-equivalent to an* $\mathcal{F}_0$*-manifold:*

$$\bar{R} = -(\mathrm{d}u(\xi)^2\bar{\pi}_1 + \tilde{\varepsilon}e^{-4u}\sqrt{\nu^2 + \tilde{\nu}^2}\left[\bar{\pi}_3 + \bar{\pi}_5\right],$$

$$\bar{\rho} = -4(\mathrm{d}u(\xi))^2\bar{g} - 2\tilde{\varepsilon}e^{-4u}\sqrt{\nu^2 + \tilde{\nu}^2}\bar{g}^*,$$

$$\bar{\tau} = -20(\mathrm{d}u(\xi))^2, \qquad \bar{\tau}^* = 8\tilde{\varepsilon}e^{-4u}\sqrt{\nu^2 + \tilde{\nu}^2}.$$

where $\tilde{\varepsilon} = \mathrm{sgn}(\tilde{\nu})$, $\bar{g}^* = \bar{g}(\cdot, \varphi\cdot)$, *and the functions* ν, $\tilde{\nu}$, $\mathrm{d}u(\xi) \neq 0$ *are point-wise constant.*

Hence, we give a geometric characterization of such manifolds in the next assertions.

Theorem 2.6 *Every 5-dimensional* $\mathcal{F}_5^0$*-manifold* $\bar{M}$ *is almost Einsteinian with negative point-wise constant scalar curvatures and it belongs to* $\omega_1 \oplus \omega_2 \oplus \omega_5 \oplus \omega_6 \oplus \omega_7 \oplus \omega_8 \oplus w_1$.

Theorem 2.7 *The 5-dimensional* $\mathcal{F}_5^0$*-manifold* $\bar{M}$ *is Einsteinian if and only if its* C_5^0*-equivalent manifold* M *is a flat* $\mathcal{F}_0$*-manifold.*

In the last case the curvature characteristics of an Einsteinian $\bar{M}$ are:

$$\bar{R} = -(\mathrm{d}u(\xi))^2\bar{\pi}_1, \qquad \bar{\rho} = -4(\mathrm{d}u(\xi))^2\bar{g}, \qquad \bar{\tau} = -20(\mathrm{d}u(\xi))^2, \qquad \bar{\tau}^* = 0.$$

Therefore we finally receive

Theorem 2.8 *The* $\mathcal{F}_5^0$*-manifold* $\bar{M}$ *has constant sectional curvatures if and only if it is Einsteinian.*

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КРИВИННИ ТЕНЗОРИ ВЪРХУ НЯКОИ ПЕТМЕРНИ ПОЧТИ КОНТАКТНИ -МЕТРИЧНИ МНОГООБРАЗИЯ

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Разгледани са 5-мерни почти контактни В-метрични многообразия от два основни класа. Доказано е, че всяко многообразие от сечението на тези класове е с точково постоянни секционни кривини. Изучен е кривинният тензор на многообразията от тези два класа и са дадени техни геометрични характеристики.