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ON SOME SUBCLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS

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The object of this paper is to obtain sharp results involving coefficient bounds, growth and distortion properties for some classes of analytic and multivalent functions in the open unit disk.

1. Introduction and definitions. Let $S(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N := \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$E := \{z : z \in C \text{ and } |z| < 1\}.$$

We denote by $(f * g)(z)$ the Hadamard product of two functions $f(z)$ and $g(z)$ in $S(p)$ where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (p \in N). \tag{1.2}$$

Thus

$$(f * g)(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.3}$$

The $(n + p - 1)$-th order Ruscheweyh derivative $D^{n+p-1} f(z)$ of a function $f(z)$ in $S(p)$ is defined by

$$D^{n+p-1} f(z) := \frac{z^p \left(z^{n-1} f(z)\right)^{n+p-1}}{(n + p - 1)!}, \tag{1.4}$$

where n is any integer such that $n > -p$. It is easy to see from (1.3) and (1.4) that

$$\begin{aligned} D^{n+p-1} f(z) &= \frac{z^p}{(1 - z)^{n+p}} * f(z) \\ &= z^p + \sum_{k=1}^{\infty} \delta(n, k) a_{p+k} z^{p+k}, \end{aligned} \tag{1.5}$$

where

$$\delta(n, k) := \binom{n + p + k - 1}{n + p - 1} = \binom{n + p + k - 1}{k} \quad (k \in N). \tag{1.6}$$

The symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z) \in S(1)$ by Al-Amiri [1].

Let $T(p)$ denote the subclass of $S(p)$ consisting of analytic and p-valent functions which can be expressed in the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0;\ p \in N). \tag{1.7}$$

Let $T_{n,p}[A, B, \alpha, \beta]$ denote the class of analytic and p-valent functions $f(z)$ belonging to the class $T(p)$ and satisfying the additional condition

$$\left| \frac{z^{-p} D^{n+p} f(z) - 1}{B z^{-p} D^{n+p} f(z) - [B + (A - B)(1 - \alpha)]} \right| < \beta \tag{1.8}$$

$$(z \in E;\ 0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1).$$

The class $T_{-p,p}[-1, 1, \alpha, \beta]$ $(0 \leq \alpha < 1;\ 0 < \beta \leq 1)$ was studied by Srivastava and Aouf [2].

The class $T_{d,1}[A, B, 0, 1]$ $(d \geq -1;\ -1 \leq A \leq 1;\ -1 \leq B \leq 0;\ B \leq A)$ was studied by Chen [3].

The class $T_{n,1}[A, B, \alpha, \beta]$ was studied by the author [4].

The class $T(1)$ is the well known class of functions with negative coefficients introduced by H. Silverman [5].

By a specialization of the parameters α, β, A, B, p, and n involved in the class $T_{n,p}[A, B, \alpha, \beta]$, we obtain some subclasses studied by various authors [2,3,4].

The object of the present paper is to show some results involving coefficient bounds, growth and distortion properties for the class $T_{n,p}[A, B, \alpha, \beta]$.

2. A theorem on coefficient bounds.

Theorem 1. *A function $f(z)$ defined by (1.7) is in the class $T_{n,p}[A, B, \alpha, \beta]$ if and only if*

$$\sum_{k=1}^{\infty} (1 + \beta B) \delta(n + 1, k) a_{p+k} \leq (B - A) \beta (1 - \alpha). \tag{2.1}$$

The result is sharp.

Proof. Let inequality (2.1) holds true and let $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left| z^{-p} D^{n+p} f(z) - 1 \right| - \beta \left| B z^{-p} D^{n+p} f(z) - [B + (A - B)(1 - \alpha)] \right| \\
&= \left| -\sum_{k=1}^{\infty} \delta(n + 1, k) a_{p+k} z^k \right| - \beta \left| (B - A)(1 - \alpha) - B \sum_{k=1}^{\infty} \delta(n + 1, k) a_{p+k} z^k \right| \\
&\leq \sum_{k=1}^{\infty} \delta(n + 1, k) a_{p+k} - \beta \left[(B - A)(1 - \alpha) - B \sum_{k=1}^{\infty} \delta(n + 1, k) a_{p+k} \right] \\
&= \sum_{k=1}^{\infty} (1 + \beta B) \delta(n + 1, k) a_{p+k} - (B - A) \beta (1 - \alpha)) \leq 0
\end{aligned}$$

by the hypothesis of Theorem 1, $\delta(n, k)$ being defined by (1.6). By the Maximum Modulus Theorem, we have $f(z) \in T_{n,p}[A, B, \alpha, \beta]$.

In order to prove the converse, we assume that $f(z)$ is defined by (1.7) and is the class $T_{n,p}[A,B,\alpha,\beta]$. Then the condition (1.8) readily yields

$$\left|\frac{z^{-p}D^{n+p}f(z)-1}{Bz^{-p}D^{n+p}f(z)-[B+(A-B)(1-\alpha)]}\right|$$

$$(2.2)\qquad =\left|\frac{-\sum\limits_{k=1}^{\infty}\delta(n+1,k)a_{p+k}z^k}{(B-A)(1-\alpha)-B\sum\limits_{k=1}^{\infty}\delta(n+1,k)a_{p+k}z^k}\right|<\beta \qquad z\in E.$$

Since $|R(z)|\le|z|$ for all z, we find from (2.2) that

$$(2.3)\qquad R\left\{\frac{\sum\limits_{k=1}^{\infty}\delta(n+1,k)a_{p+k}z^k}{(B-A)(1-\alpha)-B\sum\limits_{k=1}^{\infty}\delta(n+1,k)a_{p+k}z^k}\right\}<\beta \qquad z\in E.$$

Now we choose values of z on the real axis so that $z^{-p}D^{n+p}f(z)$ is real. So letting $z\to 1$ through real values we obtain

$$(2.4)\qquad \sum_{k=1}^{\infty}\delta(n+1,k)a_{p+k}\le(B-A)\beta(1-\alpha)-\beta B\sum_{k=1}^{\infty}\delta(n+1,k)a_{p+k},$$

which gives us the desired assertion (2.1).

The assertion (2.1) of Theorem 1 is sharp. The extremal function is

$$(2.5)\qquad f(z)=z^p-\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)\delta(n+1,k)}z^{p+k} \qquad k\in N.$$

Corollary 1. *Let the function $f(z)$ defined by (1.7) be in the class $T_{n,p}[A,B,\alpha,\beta]$. Then*

$$(2.6)\qquad a_{p+k}\le\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)\delta(n+1,k)} \qquad k\in N.$$

The equality in (2.6) is attained for the function $f(z)$ given by (2.5).

3. Growth and distortion properties.

Theorem 2. *Let the function f(z) defined by (1.7) be in the class $T_{n,p}[A,B,\alpha,\beta]$. Then for $|z|=r$ $(0<r<1)$,*

$$(3.1)\qquad r^p-\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+1+p)}r^{p+1}\le|f(z)|\le r^p+\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+1+p)}r^{p+1},$$

$$(3.2)\qquad pr^{p-1}-\frac{(B-A)\beta(1-\alpha)(p+1)}{(1+\beta B)(n+1+p)}r^p\le|f'(z)|\le pr^{p-1}+\frac{(B-A)\beta(1-\alpha)(p+1)}{(1+\beta B)(n+1+p)}r^p, s$$

$$(3.3)\qquad s1-\frac{(B-A)\beta(1-\alpha)}{1+\beta B}r\le\left|\frac{D^{n+p}f(z)}{z^p}\right|\le 1+\frac{(B-A)\beta(1-\alpha)}{1+\beta B}r.$$

Each of these results is sharp.

Proof. Since $f(z) \in T_{n,p}[A, B, \alpha, \beta]$, in view of Theorem 1, we have

$$(1 + \beta B)\delta(n + 1, 1) \sum_{k=1}^{\infty} a_{p+k} \leq \sum_{k=1}^{\infty}(1 + \beta B)\delta(n + 1, k)a_{p+k} \leq (B - A)\beta(1 - \alpha) \tag{3.4}$$

which immediately yields

$$\sum_{k=1}^{\infty} a_{p+k} \leq \frac{(B - A)\beta(1 - \alpha)}{(1 + \beta B)(n + 1 + p)}. \tag{3.5}$$

Consequently, for $|z| = r$ $(0 < r < 1)$, we obtain

$$|f(z)| \geq r^p - r^{p+1} \sum_{k=1}^{\infty} a_{p+k} \geq r^p - \frac{(B - A)\beta(1 - \alpha)}{(1 + \beta B)(n + 1 + p)} r^{p+1} \tag{3.6}$$

and

$$|f(z)| \leq r^p + r^{p+1} \sum_{k=1}^{\infty} a_{p+k} \leq r^p + \frac{(B - A)\beta(1 - \alpha)}{(1 + \beta B)(n + 1 + p)} r^{p+1}, \tag{3.7}$$

which proves the assertion (3.1) of Theorem 2.

Now, it is easily seen from (1.7) that, for $|z| = r$ $(0 < r < 1)$

$$|f'(z)| \geq pr^{p-1} - r^p \sum_{k=1}^{\infty}(p + k)a_{p+k} \tag{3.8}$$

and

$$|f'(z)| \leq pr^{p-1} + r^p \sum_{k=1}^{\infty}(p + k)a_{p+k}. \tag{3.9}$$

In view of Theorem 1, we have

$$\frac{(1 + \beta B)\delta(n + 1, 1)}{p + 1} \sum_{k=1}^{\infty}(p + k)a_{p+k} \leq \sum_{k=1}^{\infty}(1 + \beta B)\delta(n + 1, k)a_{p+k} \leq (B - A)\beta(1 - \alpha),$$

which readily yields

$$\sum_{k=1}^{\infty}(p + k)a_{p+k} \leq \frac{(B - A)\beta(1 - \alpha)(p + 1)}{(1 + \beta B)(n + 1 + p)}. \tag{3.10}$$

Upon substituting from (3.10) into the second members of (3.8) and (3.9), we obtain the assertion (3.2) of Theorem 2.

Next, by using the second inequality in (3.4) we observe that, for $|z| = r$ $(0 < r < 1)$

$$\left|z^{-p} D^{n+p} f(z)\right| \leq 1 + r \sum_{k=1}^{\infty} \delta(n + 1, k)a_{p+k} \leq 1 + \frac{(B - A)\beta(1 - \alpha)}{1 + \beta B} r \tag{3.11}$$

and

$$\left|z^{-p} D^{n+p} f(z)\right| \geq 1 - r \sum_{k=1}^{\infty} \delta(n + 1, k)a_{p+k} \geq 1 - \frac{(B - A)\beta(1 - \alpha)}{1 + \beta B} r, \tag{3.12}$$

which proves the assertion (3.3) of Theorem 2.

Equalities in (3.1), (3.2) and (3.3) of Theorem 2 are attained for the functions $f(z)$

given by

$$f(z) = z^p - \frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+1+p)} z^{p+1}, \qquad z = \pm r. \tag{3.13}$$

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ВЪРХУ НЯКОИ ПОДКЛАСОВЕ АНАЛИТИЧНИ И МНОГОЛИСТНИ ФУНКЦИИ

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Предмет на тази статия е получаването на точни резултати за коефициентите и някои оценки за някои класове аналитични и многолистни функции в единичния кръг.