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AN APPLICATION OF THE SKEW-SYMMETRIC STANILOV CURVATURE OPERATOR IN THE CLASSICAL EGOLOGY*

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The main result in this paper is that locally sprays of the Hamilton systems are distincts from the ecologically point of view if and only if a population space (M,g) is SB population space.

1. Introduction. Let (M,g) be an n-dimensional Riemannian manifold. Fixing a point $p \in M$, we can always consider the tangent space M_p to the manifold M, at this point p. If M is a Riemannian manifold and if ∇ is the Levi-Civita connection on M, then we can consider the curvature tensor R in M_p, which is a four-linear mapping satisfying the following properties:

$$R(x,y,z,u) = -R(y,x,z,u),\ R(x,y,z,u) = -R(x,y,u,z), \\ R(x,y,z,u) + R(y,z,x,u) + R(z,x,y,u) = 0, \tag{1}$$

Using these properties of R, we can define some curvature operators in the vector space M_p for example Jacobi operator, Stanilov operators and e.t.c.[1]. In the present note we will use one of the Stanilov curvature operators namely a skew-symmetric linear operator in the tangent space M_p, at a point $p \in M$, defined by $S(E^2)(u) = R(X,Y,u)$, where X,Y is an orthonormal basis in the plane $E^2 \subseteq M_p$ [2]. It is easy to see that this operator is invariant under all orthogonal transformations in E^2. The theory of this operator started by Stanilov and Ivanova which investigated four-dimensional Riemannian manifolds with a pointwice constants eigenvalues of the curvature operator $S(E^2)$, for arbitrary plane $E^2 \subseteq M_p$, at any point $p \in M$ [2]. The classes of such manifolds are called Stanilov(S) manifolds as afterwards Gilkey, Leahy, and Sadofsky proved that *any n-dimensional Riemannian S-manifold is a space with a structure of almost product P, and with a curvature tensor R of the form* [1], [3]

$$R(x,y,z) = g(Py,x)Pz - g(Px,z)Py. \tag{2}$$

We provided in Lorentzian setting analogous investigations when $\dim M = 4$, supposing that in any cases the plane E^2 is not isotropic plane. We proved

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Theorem 1 [4]. *Let (M, g) be a four-dimensional Lorentzian manifold with pointwise constant eigenvalues of the Stanilov curvature operator $S(E^2)$, for arbitrary non-isotropic plane E^2, at arbitrary point $p \in M$. Then (M, g) is a Lorentzian space of a constant sectional curvature, or (M, g) is a Lorentzian space of maximal mobility with metric (in a special coordinate system) of the form*

$$ds^2 = -sh^2 x_4 dx_2^2 - \sin^2 x_4 dx_3^2 + 2dx_1 dx_4. \tag{3}$$

2. Population spaces and Stanilov curvature operator. Let (M, g) be a population space considered as a Riemannian manifold, and let (x_i, N^i) denote the natural coordinates in a linear chart of the tangent bundle TM^n. We consider the Voltera-Hamilton systems of differential equations:

$$\begin{aligned} \frac{dx^i}{dt} &= k_{(i)} N^i \qquad \text{(not summed)}, \\ \frac{dN^i}{dt} &= -G^i_{jk} N^j N^k + r^i_j N^j + e^i, \end{aligned} \tag{4}$$

with all coefficients possibly depend on x^i, N^i, t, the quantities G^k_{ij} being n^3 functions homogenous of degree zero in N^i, and with a smooth initial conditions x^i_0, N^i_0, t_0. The coordinates x^i are Voltera production's variables whose constant per capita rate of increase is $k(i)$, while the second system of equations is a description of how different species population, $N^i \geq 0$, growth r^j_i, interact G^k_{ij} and react e^i to the external environment (and may be noisy). For almost two decades these systems (of 2^{nd} order) has played the major role in the modelling ecology and evolution of colonial organisms, which may involve chemical warfare when G^k_{ij} depend explicity on x^i, and social interactions, when G^k_{ij} depend on ratio of population densities N^i/N_j [5]. Our natural geometrical approach starts from the second order system of differential equations, obtained from (4) by $e^i = 0$ and $r_i \lambda \delta^j_i$, where λ is a constant. Then we have the local spray

$$\frac{d^2 x^i}{ds^2} + G^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{5}$$

where $ds = e^{\lambda t} dt$ and make use of the local spray connection

$$G^i_j = \nabla_j G^i, \tag{6}$$

and the linear connection

$$G^i_{jk} = \nabla_k G^i_j. \tag{7}$$

If all interaction coefficients G^i_{jk} are zero, then we say that the spary has straight line form. The spray curvature tensor(operator) B of type (1,3), defined from Berwald [5], was defined as

$$B^i_{hjk} = \nabla_k G^i_{hj} + G^r_{hj} G^i_{rk} + G^r_j D^i_{rhk} - (\nabla_j G^i_{hk} + G^r_{hk} G^i_{rj} + G^r_k D^i_{rhj}), \tag{8}$$

where D is the sectional curvature, of the corresponding to B tensor of type (0,4). Following some result in [5] we can formulate

Theorem 2. *The spray curvature operator B can be transformed into straight-line form via smooth coordinate changes if and only if the eigenvalues of this operator are equal to zero.*

Using the spray curvature tensor B we can define a skew-symmetric linear *spray* *curvature operator* $SB(E^2)u = B(x,y,u)$, where x,y is an orthonormal basis in the tangent space M_p, at the point p, of the population space M. We define the corresponding to B Ricci tensor ρ, by $\rho(x,y) = \text{trace } (u \to B(u,x,y)), x,y,u \in M_p, p \in M$.

Definition 1. *We say that population space (M,g) is spray-Einstein population space if $\rho(x,y) = kg(x,y)$, where k is a constant, $x,y \in M_p$, and $p \in M$ is an arbitrary point. We say that (M,g) is SB Stanilov-Berwald) population space if the eigenvalues of the curvature operator $SB(E^2)$ don't depend on the plane $E^2 \subset M_p$, at any point $p \in M$.*

3. An ecological problem and characterization of the four-dimensional Lorentzian SB population spaces. A Riemannian manifold (M,g) is called Lorentzian manifold if at any point $p \in M$, the tangent space M_p is equipped with inner product g of a signature $(-,+,\dots,+)$ or $(+,+,\dots,+,-)$. A tangent vector X is called spacelike or timelike if $g(X,X) = 1$ or $g(X,X) = -1$. The set of all timelike tangent vectors in M_p we denote by ${}^{-}S_pM$, respectivelly the set of all spacelike tangent vector in M_p we denote by ${}^{+}S_pM$. As in the Riemannian setting, we define second exterior product $\wedge^2(M_p)$, which is a 6-dimensional vector space, such that

$$P_1 = e_1 \wedge e_2, P_2 = e_1 \wedge e_3, P_3 = e_1 \wedge e_4, P_4 = e_2 \wedge e_3, P_5 = e_2 \wedge e_4, P_6 = e_3 \wedge e_4, \tag{9}$$

formed an orthonormal basis in $\wedge^2(M_p)$, for any orthonormal Lorentzian basis e_1, e_2, e_3, e_4 ($e_4 \in^{-} S_pM$) in $M_p, p \in m$. It was proved

Theorem 3 [6]. *Let (M,g) be a four-dimensional Einstein Lorentzian manifold. Then at any point $p \in M$, there exists a Lorentzian basis $e_1, e_2, e_3, e_4 (e_4 \in^{-} S_pM)$ in M_p, such that the matrix of the curvature operator $\mathcal{R}$ in $\wedge^2(M_p)$, with respect to the orthonormal basis (9), has the form $\left(\begin{smallmatrix} M & N \\ -N & M \end{smallmatrix}\right)$, where the matrix M and N must be of one of the following types:*

I.
$$M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$
$$\alpha_1 + \alpha_2 + \alpha_3 = \lambda, \qquad \beta_1 + \beta_2 + \beta_3 = 0;$$

II.
$$M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 + 1 & 0 \\ 0 & 0 & \alpha_2 - 1 \end{pmatrix}, N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 1 & \beta_2 \end{pmatrix},$$
$$\alpha_1 + 2\alpha_2 = \lambda, \qquad \beta_1 + 2\beta_2 = 0,$$

III.
$$M = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad 3\alpha = \lambda.$$

Using (1) and (9) we can prove

Theorem 4. *Let (M,g) be a four-dimensional spray-Einstein Lorentzian population space. Then at any point $p \in M$ there exists a Lorentzian basis $e_1, e_2, e_3, e_4 (e_4 \in^{-} S_pM)$ in M_p, with respect to which all components of the spray curvature tensor B can be expressed by only one of the following formulas:*

$$B_{1212} = -B_{3434} = \alpha_1, \; B_{1313} = -B_{2424} = \alpha_2, \; B_{2323} = -B_{1414} = \alpha_3; \tag{10}$$

$$B_{1212} = -B_{3434} = \alpha_1,\ B_{1313} = -B_{2424} = \alpha_2 + 1, \tag{11}$$
$$B_{2323} = -B_{1414} = \alpha_2 - 1, B_{3114} = -B_{3224} = 1;$$

$$B_{1212} = -B_{3434} = B_{1313} = -B_{2424} = B_{2323} = -B_{1414} = \alpha, \tag{12}$$
$$B_{3114} = -B_{3224} = 1,\ B_{2443} = -B_{2113} = 1.$$

Using this result we check that the characteristic equation of the curvature operator $SB(E^2)$, with respect to the Lorentzian basis $e_1, e_2, e_3, e_4 (e_4 \in^- S_pM)$, in the tangent space M_p, can be written in the form

$$\mu^4 - I_2(p; X, Y).\mu^2 - I_4(p; X, Y) = 0,$$

where

$$I_2(p; X, Y) = B^2_{XY12} + B^2_{XY13} + B^2_{XY23} - B^2_{XY14} - B^2_{XY24} - B^2_{XY34},$$

$$I_4(p; X, Y) = -(B_{XY12}B_{XY34} - B_{XY13}B_{XY24} + B_{XY23}B_{XY14})^2.$$

Now following the methods in [4], we can obtain that if for B holds (10), then

$$\begin{aligned} &I_2(p; e_1, e_2) = -I_2(p; e_3, e_4) = \alpha_1^2 - \beta_1^2, I_2(p; e_1, e_3) = -I_2(p; e_2, e_4) = \alpha_2^2 - \beta_2^2,\\ &I_2(p; e_2, e_3) = -I_2(p; e_1, e_4) = \alpha_3^2 - \beta_3^2; I_4(p; e_1, e_2) = -I_4(p; e_3, e_4) = -\alpha_1^2\beta_1^2,\\ &I_4(p; e_1, e_3) = -I_4(p; e_2, e_4) = -\alpha_2^2\beta_2^2, I_4(p; e_2, e_3) = -I_4(p; e_1, e_4) = -\alpha_3^2\beta_3^2. \end{aligned} \tag{10.a}$$

From our hypothesis (M, g) to be SB population space, we have

$$\begin{aligned} &I_k(p; e_1, e_2) = I_k(p; e_1, e_3) = I_k(p; e_2, e_3);\\ &I_k(p; e_1, e_4) = I_k(p; e_2, e_4) = I_k(p; e_3, e_4), \qquad k = (2, 4). \end{aligned} \tag{10.b}$$

Now from (10.a) and (10.b) we get $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_3$, which means that (M, g) is a space of constant sectional curvature.

Suppose for the components of the spray curvature tensor B holds (11). Then

$$\begin{aligned} &I_2(p; e_1, e_2) = -I_2(p; e_3, e_4) = \alpha_1^2 - \beta_1^2,\\ &I_2(p; e_1, e_3) = -I_2(p; e_2, e_4) = (\alpha_2 + 1)^2 - \beta_2^2 - 1,\\ &I_2(p; e_2, e_3) = -I_2(p; e_1, e_4) = (\alpha_2 - 1)^2 - \beta_2^2 - 1;\\ &I_4(p; e_1, e_2) = -I_4(p; e_3, e_4) = -\alpha_1^2\beta_1^2,\\ &I_4(p; e_1, e_3) = -I_4(p; e_2, e_4) = -\beta_2^2(\alpha_2 + 1)^2,\\ &I_4(p; e_2, e_3) = -I_4(p; e_1, e_4) = -\beta_2^2(\alpha_2 - 1)^2. \end{aligned} \tag{11.a}$$

From here and (10.b), we get that if (M, g) is SB population space, then $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 0$, which is necessary and sufficient condition (M, g) to be a Lorentzian space of maximal mobility with metric of the form (3)[6].

Finally we remark that if for B holds (12), then $\alpha = 0$, which is impossible [6].

Thus we can formulate

Theorem 5. *Let (M, g) be a four-dimensional spray-Einstien Lorentzian population space. Then (M, g) is SB-population space if and only if (M, g) is a space of constant sectional curvature or (M, g) is a space of maximal mobility with metric of the form* (3)*.*

From this theorem and the main results in [5] we get our main result

Theorem 6. *Let (M, g) be a four-dimensional spray-Einstein Lorentzian population space which is not with a constant sectional curvature. Then the Voltera-Hamilon systems and their local sprays have straight line forms and are distinct from the ecologically point of view, if and only if (M, g) is Stanilov-Berwald population space.*

Finally we denote that some similarly results was provided in marine ecology [7].

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ПРИЛОЖЕНИЕ НА АНТИСИМЕТРИЧНИЯ ОПЕРАТОР НА СТАНИЛОВ В КЛАСИЧЕСКАТА ЕКОЛОГИЯ

Веселин Видев, Мария Вълкова, Йордан Стайков

Главен резултат в статията е, че локалните вълни (следствия) от системата на Хамилтон са различни от екологична гледна точка тогава и само тогава, когато популационното пространство (M, g) е SB пространство.