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WEYL SPACES OF COMPOSITIONS

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Three Weyl spaces of nonorthogonal compositions of two base manifolds are investigated in the present paper. The hybridian tensor generated by the first Weyl space and introduced by G. Timofeev is chosen as a fundamental tensor for the second of them and the third Weyl space is conformal to the second one. Necessary and sufficient conditions, so that these three Weyl spaces to be spaces of compositions are found with the help of the prolonged covariant differentiation. The relations between the tensors of curvatures of these three Weyl spaces of compositions are established. The characteristics for an isotropic composition in these spaces are obtained.

1. Preliminary. Let $W_N(g_{\alpha\beta}, T_\sigma)$ be Weyl space with a fundamental tensor $g_{\alpha\beta}$ and a complementary covector T_σ.

After renormalization of the fundamental tensor by the law

$$\breve{g}_{\alpha\beta} = \lambda^2 g_{\alpha\beta}, \tag{1}$$

where λ is a point function, the complementary covector T_σ transforms by the law ([1], p.152)

$$\breve{T}_\sigma = T_\sigma + \partial_\sigma ln\lambda. \tag{2}$$

According to ([1], p.152) the fundamental tensor $g_{\alpha\beta}$ and the complementary covector T_σ satisfy the identities

$$\nabla_\sigma \; g_{\alpha\beta} = 2T_\sigma \; g_{\alpha\beta}, \; \nabla_\sigma \; g^{\alpha\beta} = -2T_\sigma \; g^{\alpha\beta} \tag{3}$$

where $g^{\alpha\beta}$ is the reciprocal tensor to $g_{\alpha\beta}$ and

$$g_{\alpha\beta} \; g^{\alpha\sigma} = \delta^\sigma_\beta. \tag{4}$$

The equalities

$$\overset{\circ}{\nabla}_\sigma \;\; g_{\alpha\beta} = 0, \; \overset{\circ}{\nabla}_\sigma \;\; g^{\alpha\beta} = 0, \tag{5}$$

where with $\overset{\circ}{\nabla}$ is denoted the prolonged covariant differentiation, are introduced in [8].

Consider in the space W_N the composition $X_n \times X_m$ of two base manifolds X_n and $X_m, (n+m=N)$. The spaces W_N of compositions will be denoted $W_N(X_n \times X_m)$. Two positions $P(X_n)$ and $P(X_m)$ of the base manifolds pass through any point of the space $W_N(X_n \times X_m)$.

According to [2] and [3] each composition is completely definite with the field of the affinor a_α^β, satisfying the condition

$$a_\alpha^\sigma a_\sigma^\beta = \delta_\alpha^\beta. \tag{6}$$

The affinor a_α^β is called an affinor of the composition.

We presuppose that the composition is always integrable and nonorthogonal. The condition for integrability of the structure characterizes with the equality [6] and

$$a_\beta^\sigma \nabla_{[\alpha}\ a_{\sigma]}^\nu - a_\alpha^\sigma \nabla_{[\beta}\ a_{\sigma]}^\nu = 0. \tag{7}$$

According to [6, 7] the projecting affinors $\overset{n}{a}{}_\alpha^\beta$, $\overset{m}{a}{}_\alpha^\beta$ are defined by the equalities

$$\overset{n}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta), \quad \overset{m}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta). \tag{8}$$

It is true the equality $g_{\alpha\beta} = \overset{n}{G}_{\alpha\beta} + 2g_{\alpha\beta} + \overset{m}{G}_{\alpha\beta}$, where

$$\overset{n}{G}_{\alpha\beta} = \overset{n}{a}{}_\alpha^\sigma\ \overset{n}{a}{}_\beta^\nu\ g_{\sigma\nu}, \ \overset{m}{G}_{\alpha\beta} = \overset{m}{a}{}_\alpha^\sigma\ \overset{m}{a}{}_\beta^\nu\ g_{\sigma\nu}, \ G_{\alpha\beta} = a_{(\alpha}^\sigma\ a_{\beta)}^\nu\ g_{\sigma\nu}. \tag{9}$$

The tensor $G_{\alpha\beta}$ is named hybridian metrical tensor [4]. Due of (5) $G_{\alpha\beta} \neq 0$ for nonorthogonal compositions $X_n \times X_m$.

Acccording to [4] the tensor of the composition

$$a_{\alpha\beta} = a_\alpha^\sigma\ g_{\sigma\beta} \tag{10}$$

and the fundamental tensor of W_N satisfy the equation

$$a_{\alpha\sigma}\ g^{\sigma\nu}\ a_{\nu\beta} = g_{\alpha\beta}. \tag{11}$$

In [4] Timofeev has proved the following theorem

Theorem 1. *The Weyl space $W_N(g_{\alpha\beta}, T_\sigma)$ is a space of composition $W_N(X_n \times X_m)$, if and only if there exists a tensor $a_{\alpha\beta}$, satisfying equalities* (11) *and*

$$g^{\sigma\rho}\ a_{\beta\rho}\ \nabla_{[\alpha}a_{\sigma]\nu} - g^{\sigma\rho}\ a_{\alpha\rho}\ \nabla_{[\beta}a_{\sigma]\nu} - 2T_{[\alpha}\ g_{\beta]\nu} - 2T_\sigma g^{\sigma\rho}\ a_{[\alpha/\rho/}\ a_{\beta]\nu} = 0. \tag{12}$$

2. Weyl spaces of compositions. Let us note that by means of the prolonged covariant differentiation the equality (12) can be written in the form

$$g^{\sigma\rho}\ a_{\beta\rho}\ \overset{\circ}{\nabla}_{[\alpha}a_{\sigma]\nu} - g^{\sigma\rho}\ a_{\alpha\rho}\ \overset{\circ}{\nabla}_{[\beta}a_{\sigma]\nu} = 0. \tag{13}$$

We denote by ∇ and ${}^a\nabla$ covariant derivatives in Weyl spaces W_N and ${}^aW_N, a = 1, 2$, respectively.

Let $W_N(g_{\alpha\beta}, T_\sigma)$ be a space of composition $W_N(X_n \times X_m)$.

Theorem 2. *The Weyl space* ${}^1W_N(G_{\alpha\beta}, T_\sigma)$ *is a space of composition* ${}^1W_N(X_n \times X_m)$*, if and only if there exists a tensor* $a_{\alpha\beta}$*, satisfying equalities*

$$a_{\sigma\alpha}\ G^{\sigma\nu}\ a_{\beta\nu} = G_{\alpha\beta}, \tag{14}$$

$$G^{\sigma\rho}\ a_{\rho\beta}\ {}^1\overset{\circ}{\nabla}_{[\alpha} a_{/\nu/\sigma]} - G^{\sigma\rho}\ a_{\rho\alpha}\ {}^1\overset{\circ}{\nabla}_{[\beta} a_{/\nu/\sigma]} = 0. \tag{15}$$

Proof. It is easily to see that the weight of the hybridian tensor $G_{\alpha\beta}$ is $\{2\}$. Hence after renormalization (1) of $g_{\alpha\beta}$ for $G_{\alpha\beta}$ we have $\breve{G}_{\alpha\beta} = \lambda^2 G_{\alpha\beta}$. Since T_σ transforms by the law (2), then there exists only one Weyl space ${}^1W_N(G_{\alpha\beta}, T_\sigma)$ with a fundamental tensor $G_{\alpha\beta}$ and a complementary covector T_σ, satisfying ${}^1\nabla_\sigma\ G_{\alpha\beta} = 2T_\sigma\ G_{\alpha\beta}$.

According to [8]

$${}^1\overset{\circ}{\nabla}_\sigma\ \ G_{\alpha\beta} = 0,\ {}^1\overset{\circ}{\nabla}_\sigma\ \ G^{\alpha\beta} = 0. \tag{16}$$

Consider the tensor

$$G^{\alpha\beta} = a_\sigma^{(\alpha}\ a_\nu^{\beta)}\ g^{\sigma\nu}. \tag{17}$$

Using (6) and (7) we find $G_{\alpha\beta}G^{\alpha\sigma} = \delta_\beta^\sigma$, which means that $G^{\alpha\beta}\{-2\}$ is a reciprocal tensor of the tensor $G_{\alpha\beta}$.

Let the Weyl space ${}^1W_N(G_{\alpha\beta}, T_\sigma)$ be a space of composition ${}^1W_N(X_n \times X_m)$. Consider the tensor of the composition $X_n \times X_m$

$${}^1a_{\alpha\beta} = a_\alpha^\sigma\ G_{\sigma\beta}. \tag{18}$$

Obviously ${}^1a_{\alpha\beta}\{2\}$. Taking into account (9), (10), (18) we obtained

$${}^1a_{\alpha\beta} = a_{\beta\alpha}. \tag{19}$$

To prove (14) it is sufficient to substitute $G^{\alpha\beta}$ and $a_{\alpha\beta}$ from (17) and (19) in (14) and keeping in mind (4) and (18).

After prolonged covariant differentiation of (18) and applying (16) and (19) we find ${}^1\overset{\circ}{\nabla}_\nu\ G_{\beta\alpha} = \nabla_\nu\ \ a_\alpha^\sigma\ G_{\sigma\beta}$. Now after substitution of $\nabla_\nu\ a_\alpha^\sigma = {}^1\overset{\circ}{\nabla}_\nu G_{\beta\alpha}\ G^{\sigma\beta}$ in (7) we obtain (15).

Theorem 3. *The Weyl space* ${}^2W_N(g_{\alpha\beta}, a_\sigma^\nu T_\nu)$ *is a space of composition* ${}^2W_N(X_n \times X_m)$*, if and only if there exists a tensor* $a_{\alpha\beta}$*, satisfying equalities* (11) *and*

$$g^{\sigma\rho}\ a_{\rho\beta}\ {}^2\overset{\circ}{\nabla}_{[\alpha} a_{\sigma]\nu} - g^{\sigma\rho}\ a_{\alpha\rho}\ {}^2\overset{\circ}{\nabla}_{[\beta} a_{\sigma]\nu} = 0. \tag{20}$$

Proof. Let us consider the covector $P_\alpha = a_\alpha^\beta T_\beta$ and the equality

$${}^2\nabla_\sigma\ g_{\alpha\beta} = 2P_\sigma\ g_{\alpha\beta}. \tag{21}$$

From (1) and (21) it follows $^2\nabla_\sigma \; \breve{g}_{\alpha\beta} = 2(\partial_\sigma \lambda + a^\nu_\sigma P_\nu) \; \breve{g}_{\alpha\beta}$, which means that P_α transforms by the law (2). Hence there exists only one weyl space $^2W_N(g_{\alpha\beta}, P_\sigma)$ with a fundamental tensor $g_{\alpha\beta}$ and a complementary covector P_σ.

According to [8] the equality (21) takes the form

$$(22) \qquad {}^2\overset{\circ}{\nabla}_\sigma \; g_{\alpha\beta} = 0.$$

Let the Weyl space $^2W_N(g_{\alpha\beta}, P_\sigma)$ be a space of composition $^2W_N(X_n \times X_m)$. It is easy to see that the tensor (10) is the tensor of the composition $X_n \times X_m$ and it satisfies (11). From (7), (10), (22) it follows (20).

3. Transformation of Weyl connectedness in $\boldsymbol{W_N(X_n \times X_m)}$.

Theorem 4. *The tensors of curvatures $R_{\alpha\beta\gamma}{}^{\cdot\sigma}_{\cdot}$, $^1R_{\alpha\beta\gamma}{}^{\cdot\sigma}_{\cdot}$, $^2R_{\alpha\beta\gamma}{}^{\cdot\sigma}_{\cdot}$ of the spaces $W_N(X_n \times X_m)$, $^1W_N(X_n \times X_m)$, $^2W_N(X_n \times X_m)$, respectively satisfy the equalities:*

$$(23) \qquad {}^1R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot} = R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot},$$

$$(24) \qquad {}^2R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot} - R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot} = 4N\nabla_{[\alpha}\overset{m}{T}_{\beta]},$$

where

$$(25) \qquad \overset{m}{T}_\beta = \overset{m}{a}{}^{\sigma}_{\beta} \; T_\sigma.$$

Proof. From $^1\nabla_\alpha \; T_\beta - \nabla_\alpha \; T_\beta = -T^\sigma_{\alpha\beta} \; T_\sigma$, where $T^\sigma_{\alpha\beta}$ is the affine strain tensor, we obtain

$$(26) \qquad {}^1\nabla_{[\alpha} \; T_{\beta]} = \nabla_{[\alpha} \; T_{\beta]}$$

According to ([1], p.157) we have

$$(27) \qquad R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot} = -2N\nabla_{[\alpha}T_{\beta]}, \; {}^1R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot} = -2N \; {}^1\nabla_{[\alpha}T_{\beta]}.$$

From (26), (27) it follows (23).

Obviously there exists a conformal maping between the spaces $W_N(X_n \times X_m)$, $^2W_N(X_n \times X_m)$. Acccording to ([1], p.157) the tensors of curvatures of the spaces $W_N(X_n \times X_m)$, $^2W_N(X_n \times X_m)$ satisfy the equality

$$(28) \qquad {}^2R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot} = R_{\alpha\beta\sigma}{}^{\cdot\sigma}_{\cdot} + 2P^{\sigma\nu}_{\cdot\cdot\gamma[\beta} \; p_{\alpha]\nu},$$

where

$$(29) \qquad 2P^{\nu\sigma}_{\cdot\cdot\gamma\beta} = \delta^\nu_\gamma \; \delta^\sigma_\beta \; + \delta^\nu_\beta \; \delta^\sigma_\gamma - g^{\nu\sigma} g_{\gamma\beta},$$

$$(30) \qquad p_{\alpha\nu} = \nabla_\alpha p_\nu - \frac{1}{2} P^{\beta\sigma}_{\cdot\cdot\alpha\nu} \; p_\beta p_\sigma,$$

(31) $$p_\alpha = T_\alpha - P_\alpha.$$

From (8), (25) and (31) we obtain $p_\alpha = 2 \overset{m}{a}{}_\alpha^\sigma T_\sigma = 2 \overset{m}{T}_\alpha$, i.e. $p_\alpha \in P(X_m)$. Takig into account (30) and $p_\alpha = 2 \overset{m}{T}_\alpha$, $P^{\nu\sigma}_{\cdot\cdot[\gamma\beta]} = 0$ we find

(32) $$p_{[\alpha\nu]} = 2\nabla_{[\alpha}\overset{m}{T}_{\nu]}.$$

At last from (28) and (32) it follows the validity of (24).

4. Isotropic composition in $W_N(g_{\alpha\beta}, T_\sigma)$. It is known that the Weyl space $W_N(g_{\alpha\beta}, T_\sigma)$ assumes an isotropic composition if and only if the tensor of the composition $a_{\alpha\beta}$ is asymmetric [4]. In this case the positions $P(X_n)$ and $P(X_m)$ have the same dimension and the space $W_N(g_{\alpha\beta}, T_\sigma)$ is even-dimensional.

Theorem 5. *The Weyl space $W_N(g_{\alpha\beta}, T_\sigma)$ assumes an isotropic composition if and only if for any field of directions v^α the fields of directions v^α and $a_\sigma^\alpha\ v^\sigma$ are orthogonal in $W_N(g_{\alpha\beta}, T_\sigma)$.*

Proof. The fields of directions v^α and $a_\sigma^\alpha\ v^\sigma$ are orthogonal in $W_N(g_{\alpha\beta}, T_\sigma)$ if and only if

(33) $$g_{\alpha\beta}\ v^\alpha\ a_\sigma^\alpha\ v^\sigma\ =\ 0.$$

According to (10) the equality (33) is equivalent to the equality $a_{\alpha\beta}\ v^\alpha\ v^\beta = 0$. Since v^α are arbitrary, then (33) is equivalent to $a_{(\alpha\beta)} = 0$, i.e. the tensor of the composition is asymmetric.

Corollary 1. *If one of the spaces of compositions $W_N(X_n \times X_m)$, ${}^1W_N(X_n \times X_m)$, ${}^2W_N(X_n \times X_m)$ assumes an isotropic composition then the other two spaces assume isotropic compositions too.*

The validity of the corollary follows from the fact that the tensors of the compositions of the spaces $W_N(X_n \times X_m)$, ${}^1W_N(X_n \times X_m)$, ${}^2W_N(X_n \times X_m)$ are $a_{\alpha\beta}$, $a_{\beta\alpha}$, $a_{\alpha\beta}$, respectively.

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ВАЙЛОВИ ПРОСТРАНСТВА ОТ КОМПОЗИЦИИ

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Нека е дадено вайлово пространство от неортогонална композиция. Разглеждат се още две вайлови пространства, свързани с даденото по следния начин: основният тензор на едното е въведеният от Тимофеев хибриден тензор, а другото вайлово пространство е конформно на даденото. С помоща на продължено ковариантно диференциране са намерени необходими и достатъчни условия трите вайлови пространства да бъдат пространства от композиции и са намерени връзки между тензорите на кривината на тези пространства. Получена е характеристика за изотопна композиция в тези пространства.