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# SOLUTIONS AND PERTURBATION THEORY OF A SPECIAL MATRIX EQUATION I: PROPERTIES OF SOLUTIONS*

**Vejdi Ismailov Hasanov, Ivan Ganchev Ivanov**

*Dedicated to Milko Petkov on the occasion of his 70th birthday*

A special nonlinear matrix equation is considered. Theorems for the existence of a special positive definite solution $X_l$ and a minimal positive definite solution $X_S$ are proved. Some estimates of these solutions are derived.

**1. Introduction.** We consider the nonlinear matrix equation

$$X + A^* X^{-n} A = Q\,, \tag{1}$$

where $A$, $Q \in C^{m\times m}$, and $Q$ is a positive definite matrix, and $n$ is an integer. We study the properties of positive definite solutions of equation (1). The more general equation $X + A^*\mathcal{F}(X)A = Q$ has been investigated in [2, 5]. Some necessary and sufficient conditions for the existence of a solution are derived in [5]. The Hermitian positive definite solutions of the equation (1) and its properties have been studied in [3]. The consideration of the nonlinear matrix equations $X \pm A^*X^{-n}A = Q$ and a perturbation theory for this equation is proposed in [4]. In this paper we continue to investigate properties of the solutions of the equation (1). Properties of the positive definite solutions of equation (1) with $n = 1$ are studied in [6].

Let $X_S$ and $X_L$ be positive definite solutions of the equation (1). If every positive definite solution $X$ satisfies $X_S \le X \le X_L$, then $X_S$ and $X_L$ are minimal and maximal solutions of (1), respectively.

We use the following notations. With $A > 0\,(A \ge 0)$ we denote a positive definite (semidefinite) matrix $A$. If $A - B > 0\,(A - B \ge 0)$ we write $A > B\,(A \ge B)$. The norm used in this paper is the spectral matrix norm.

**2. Solutions of $X + A^*X^{-n}A = Q$.**

**Theorem 1.** *If equation (1) has a positive definite solution $X$, then*

$$\sqrt[n]{AQ^{-1}A^*} \;<\; X \;\le\; Q - \frac{1}{\left(\|Q\|\,\|Q^{-1}\|\right)^{n-1}}A^*Q^{-n}A\,.$$

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**Proof.** According to Theorem 4 [4] we have $\sqrt[n]{AQ^{-1}A^*} < X$. Since $X \leq Q$, then $X^n \leq \left(\frac{M_Q}{m_Q}\right)^{n-1} Q^n$, where $m_Q\, I \leq Q \leq M_Q\, I$ [1] ($m_Q = \|Q^{-1}\|^{-1}$ and $M_Q = \|Q\|$). Hence

$$X = Q - A^* X^{-n} A \leq Q - \frac{1}{\left(\|Q\|\,\|Q^{-1}\|\right)^{n-1}}\, A^* Q^{-n} A. \qquad \square$$

**Corollary 2.** *If equation* (1) *has a positive definite solution, then*

$$Q - \frac{1}{\left(\|Q\|\,\|Q^{-1}\|\right)^{n-1}} A^* Q^{-n} A - \sqrt[n]{AQ^{-1}A^*} > 0\,.$$

### 2.1. A Special Solution $X_l$.

**Theorem 3.** *If* $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$, *then there exists a positive definite solution* $X_l$ *of the matrix equation* (1) *for which the inequalities*

$$\text{(2)} \qquad \frac{n}{(n+1)\|Q^{-1}\|}\, I < X_l \leq Q - \frac{1}{\left(\|Q\|\,\|Q^{-1}\|\right)^{n-1}} A^* Q^{-n} A$$

*hold. The inequalities* (2) *are satisfied for the solution* $X_l$ *only.*

**Proof.** Consider the recurrence equation

$$\text{(3)} \qquad X_{k+1} = Q - A^* X_k^{-n} A\,, \quad X_0 = \gamma I,$$

where $\gamma \in \left(\frac{n}{(n+1)\|Q^{-1}\|},\, \frac{1}{\|Q^{-1}\|}\right]$. We will show that $\frac{n}{(n+0)\|Q^{-1}\|}\, I < X_k \leq Q$ for every matrix $X_k$ of the matrix sequence (3). Since $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$ we have $A^*A < \frac{n^n}{\left[(n+1)\|Q^{-1}\|\right]^{n+1}}\, I$. Consider the function $\varphi(\alpha) = \alpha^n\left(\frac{1}{\|Q^{-1}\|} - \alpha\right)$. We have

$$\max_{\alpha>0} \varphi(\alpha) = \varphi\left(\frac{n}{(n+1)\|Q^{-1}\|}\right) = \frac{n^n}{\left[(n+1)\|Q^{-1}\|\right]^{n+1}}\,.$$

The function $\varphi$ is continuous and monotone decreasing on $\left(\frac{n}{(n+1)\|Q^{-1}\|},\, \frac{1}{\|Q^{-1}\|}\right]$. Hence for every $\gamma \in \left(\frac{n}{(n+1)\|Q^{-1}\|},\, \frac{1}{\|Q^{-1}\|}\right]$ there exists $\alpha_0 \in \left(\frac{n}{(n+1)\|Q^{-1}\|},\, \gamma\right]$, such that $A^*A \leq \alpha_0^n \left(\frac{1}{\|Q^{-1}\|} - \alpha_0\right) I\,.$

We have $X_0 = \gamma I \geq \alpha_0 I$.

Assume that $X_k \geq \alpha_0 I$. We have $X_k^{-n} \leq \dfrac{1}{\alpha_0^n} I$. For $X_{k+1}$ we find

$$\begin{aligned} X_{k+1} &= Q - A^* X_k^{-n} A \geq \frac{1}{\|Q^{-1}\|}\, I - \frac{A^*A}{\alpha_0^n} \\ &\geq \frac{1}{\|Q^{-1}\|}\, I - \frac{\alpha_0^n (1-\alpha_0\,\|Q^{-1}\|)}{\alpha_0^n\,\|Q^{-1}\|}\, I = \alpha_0 I \end{aligned}$$

Hence $X_k \geq \alpha_0 I$ for every $k = 1, 2, \ldots$. Obviously $X_k \leq Q$. We obtain

$$\frac{n}{(n+1)\|Q^{-1}\|}\, I < \alpha_0\, I \leq X_k \leq Q\,, \quad k = 0, 1, 2, \ldots$$

We prove that $\{X_k\}$ is a Cauchy sequence. Since

$$\begin{aligned} X_{k+1} - X_k &= A^* X_k^{-n} (X_k^n - X_{k-1}^n) X_{k-1}^{-n} A \\ &= A^* \sum_{i=1}^{n} X_k^{-i} (X_k - X_{k-1}) X_{k-1}^{i-(n+1)} A, \end{aligned}$$

then we obtain

$$\begin{aligned} \|X_{k+1} - X_k\| &\leq \|A\|^2 \sum_{i=1}^{n} \|X_k^{-i}\|\, \|X_{k-1}^{i-(n+1)}\|\, \|X_k - X_{k-1}\| \\ &\leq \frac{n\|A\|^2}{\alpha_0^{n+1}}\, \|X_k - X_{k-1}\| \leq \ldots \\ &\leq \left[\frac{n\|A\|^2}{\alpha_0^{n+1}}\right]^k \|X_1 - X_0\| = q^k \|X_1 - X_0\|\,, \end{aligned}$$

where $q = \dfrac{n\|A\|^2}{\alpha_0^{n+1}} < 1$, due to $\alpha_0 > \dfrac{n}{(n+1)\|Q^{-1}\|}$ and $\|A\|^2 < \dfrac{n^n}{\big[(n+1)\|Q^{-1}\|\big]^{n+1}}$.

Hence the sequence $\{X_k\}$ forms a Cauchy sequence in the Banach space and this sequence converges to $X_l$ and $\dfrac{n}{(n+1)\|Q^{-1}\|}\, I < X_l \leq Q$.

From Theorem 1 we have $X_l \leq Q - \dfrac{1}{\big(\|Q\|\,\|Q^{-1}\|\big)^{n-1}} A^* Q^{-n} A\,$.

Assume that there exist two solutions $X'$, $X''$ of the equation (1), such that $\dfrac{n}{(n+1)\|Q^{-1}\|}\, I < X' \leq Q$ and $n(n+1)\|Q^{-1}\|\, I < X'' \leq Q$. Then

$$\begin{aligned} \|X' - X''\| &\leq \|A\|^2 \sum_{i=1}^{n} \|(X')^{-i}\|\, \|(X'')^{i-(n+1)}\|\, \|X' - X''\| \\ &< n\|A\|^2 \left(\frac{(n+1)\|Q^{-1}\|}{n}\right)^{n+1} \|X' - X''\| < \|X' - X''\|\,. \end{aligned}$$

Hence $X' \equiv X''$. $\square$

**Remark 1.** If $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\dfrac{n^n}{(n+1)^{n+1}}}$ and if the equation (1) has a maximal positive definite solution $X_L$, then $X_L \equiv X_l$. We know $X_L \geq X_l$. Hence the inequalities (2) are satisfied for $X_L$. Using the Theorem 3 we have $X_L \equiv X_l$.

**Corollary 4.** *If* $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$, *then the solution* $X_l$ *of the matrix equation (1) satisfies the inequality*

$$\|X_l^{-1}\| < \frac{n+1}{n}\|Q^{-1}\|\,.$$

**Corollary 5.** *If* $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$, *then the solution* $X_l$ *of the matrix equation (1) satisfies the inequality* $\frac{n}{n+1}\|Q\| < \|X_l\|$.

**Proof.** Note that from Corollary 4 we obtain $\|X_l^{-1}\| < \frac{n+1}{n}\|Q^{-1}\|$. Hence

$$\begin{aligned}
\|X_l\| &= \|Q - A^*X_l^{-n}A\| \geq \|Q\| - \|A\|^2\|X_l^{-n}\| \\
&> \|Q\| - \|A\|^2\left(\frac{n+1}{n}\|Q^{-1}\|\right)^n \\
&> \|Q\| - \frac{n^n}{\big[(n+1)\|Q^{-1}\|\big]^{n+1}}\left(\frac{n+1}{n}\|Q^{-1}\|\right)^n \\
&= \|Q\| - \frac{1}{(n+1)\|Q^{-1}\|} \geq \|Q\| - \frac{1}{n+1}\,\|Q\|\,.
\end{aligned}$$

□

### 2.2. The Minimal Solution $X_S$.

**Theorem 6.** *If the matrix equation (1) with nonsingular matrix* $A$ *has a positive definite solution, then it has a minimal solution* $X_S$*. Moreover, the iterative algorithm*

$$X_{k+1} = \sqrt[n]{A(Q-X_k)^{-1}A^*}\,, \quad X_0 = \sqrt[n]{AQ^{-1}A^*} \tag{4}$$

*converges to* $X_S$.

**Proof.** The proof is analogous to the proof of the Theorem 6 [3]. It is easy to prove that the matrix sequence (4) is increasing and bounded by any positive definite solution of (1). Hence this sequence converges to the minimal solution $X_S$. □

**Theorem 7.** *If* $\|A\|\sqrt{\|Q^{-1}\|^{n+1}} \leq \sqrt{\frac{n^n}{(n+1)^{n+1}}}$, *then* $X_S$ *satisfies the inequality*

$$X_S \leq \frac{n}{(n+1)\|Q^{-1}\|}\,I\,.$$

**Proof.** According to Theorem 3 and Theorem 6 it follows that there exists $X_S$. Consider the iterative equation (4) with $X_0 = \frac{n}{(n+1)\|Q^{-1}\|}\,I$. Using induction it is easy to prove that the matrix sequence $\{X_k\}$ is monotonically decreasing and bounded from below. Hence this sequence converges to a solution $X$ with $X \leq \frac{n}{(n+1)\|Q^{-1}\|}\,I$. □

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Ivan Ganchev Ivanov
Faculty of Economics and
Business Administration
Sofia University,
1113 Sofia, Bulgaria
e-mail i_ivanov@feb.uni-sofia.bg

Vejdi Ismailov Hasanov
Laboratory of Mathematical Modelling
Shoumen University
9712 Shoumen, Bulgaria
e-mail v.hasanov@fmi.shu-bg.net

## РЕШЕНИЯ И ПЕРТУРБАЦИОННА ТЕОРИЯ ЗА СПЕЦИАЛНО МАТРИЧНО УРАВНЕНИЕ I: СВОЙСТВА НА РЕШЕНИЯТА

**Вежди Исмаилов Хасанов, Иван Ганчев Иванов**

Разгледано е едно специално нелинейно матрично уравнение. Доказани са теореми за съществуване на специално положително дефинитно решение $X_l$ и минималното положително дефинитно решение $X_S$. Получени са неравенства за тези решения.