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ON A SPECIAL POSITIVE DEFINITE SOLUTION OF A CLASS OF NONLINEAR MATRIX EQUATIONS *

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We consider the matrix equation $X + A^*X^{-n}A = I$. A new iterative method for computing a special positive definite solution of this equation is derived. Numerical experiments with two different methods for computing this solution are executed and results are compared.

1. Introduction. We consider the nonlinear matrix equation

$$X + A^*X^{-n}A = I\,, \tag{1}$$

where $A, \in C^{m\times m}$, and I is the identity matrix, and n is a positive integer. This equation and the properties of its positive definite solutions have been explored and commented on in [1]. Iterative methods for computing special positive definite solutions of (1) have been suggested in [1].

The equation $X + A^*X^{-2}A = I$ is considered in [2] and an iterative method for computing a special positive definite solution X_l is described. That solution such that the matrix X_l^{-1} has the smallest spectral norm $(\|H\| = \sqrt{\rho(H^*H)}$, where $\rho(H^*H)$ is the special radius of $H^*H)$. If P and Q are Hermitian and $P - Q$ is a positive definite matrix we write $P > Q$. For any solution X of (1) we have $X_S \le X \le X_L$, where X_S and X_L are the minimal and maximal solution, respectively.

In this study we will summarize the method for finding the positive definite solution X_l of equation (1), suggested in [2]. We will make some numerical experiments for computing X_l with the here suggested method and the iterative method, shown in [1]. We will compare the results of the experiments of the two methods.

2. New iterative method. In this section we provide an iterative algorithm for computing a positive definite solution of the equation $X + A^*X^{-n}A = I$. We will prove that the algorithm is convergent towards the solution X_l, for which the inverse matrix has the smallest spectral norm.

Lemma 1. *The function*

$$f(v) = \frac{v}{(1+v)^{n+1}},$$

defined for $v \ge 0$ *is monotonously increasing for* $v \in \left[0, \frac{1}{n}\right]$ *and monotonously decreasing for* $v \ge \frac{1}{n}$. *When* $v = \frac{1}{n}$, $f_{\max} = f\left(\frac{1}{n}\right) = \frac{n^n}{(n+1)^{n+1}}$.

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Theorem 1. *If* $\|A\| < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$, *then the solution* X_l *of the equation* (1) *satisfies*

$$\|X_l^{-1}\| < \frac{n+1}{n}.$$

Moreover, for any other positive definite solution X *we have*

$$\|X^{-1}\| \geq \frac{n+1}{n}.$$

Proof. This theorem is a corollary of the Theorem 5 in [1]. Here we will propose a new proof and a new iterative method for computing the solution X_l.

It is easy to verify that X is a solution of (1) if and only if $Y = X^{-1}$ is a solution of the equation

$$Y = A^*Y^nAY + I\,, \tag{2}$$

We define the matrix sequence $\{Y_k\}$:

$$Y_0 = I, \quad Y_{k+1} = A^*Y_k^nAY_k + I, \quad k = 0, 1, 2, \ldots. \tag{3}$$

By induction it is easy to verify that

$$\|Y_k\| \leq 1 + v_k, \quad k = 0, 1, 2, \ldots \tag{4}$$

where

$$v_0 = 0, \quad v_{k+1} = \|A\|^2(1 + v_k)^{n+1} \quad k = 0, 1, 2, \ldots. \tag{5}$$

Using the assumption $\|A\| < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$ we get

$$\frac{1}{n} > v_k > v_{k-1} \geq 0 \quad k = 1, 2, \ldots. \tag{6}$$

We have $v_0 = 0 < \frac{1}{n}$ and $v_1 = \|A\|^2 < \frac{n^n}{(n+1)^{n+1}} < \frac{1}{n}$. Further, assume that the inequalities (6) are true for $k = p$. Then

$$\begin{aligned} v_{p+1} &= \|A\|^2(1 + v_p)^{n+1} < \frac{n^n}{(n+1)^{n+1}}\left(1 + \frac{1}{n}\right)^{n+1} = \frac{1}{n}\,, \quad \textit{and} \\ v_{p+1} - v_p &= \|A\|^2\left[(1 + v_p)^{n+1} - (1 + v_{p-1})^{n+1}\right] \\ &= \|A\|^2(v_p - v_{p-1})\sum_{i=0}^{n}(1 + v_p)^{n-i}(1 + v_{p-1})^i > 0. \end{aligned}$$

Therefore, there exists a number v , such that $v \in \left(0, \frac{1}{n}\right]$ and $v = \lim_{k\to\infty} v_k$. Then from (5) we get

$$v = \|A\|^2(1 + v)^{n+1}.$$

Applying Lemma 1 and the condition $\|A\|^2 < \frac{n^n}{(n+1)^{n+1}}$, we obtain $0 < v < \frac{1}{n}$.

Consider

$$\begin{aligned} Y_{k+1} - Y_k &= A^*Y_k^n A Y_k - A^* Y_{k-1}^n A Y_{k-1} \\ &= A^* Y_k^{n-1}(Y_k - Y_{k-1})AY_k \,+\, A^*(Y_k^{n-1} - Y_{k-1}^{n-1})Y_{k-1}AY_k \\ &\quad + A^* Y_{k-1}^n A(Y_k - Y_{k-1}) \\ &= A^* \sum_{i=1}^{n} Y_k^{n-i}(Y_k - Y_{k-1}) Y_{k-1}^{i-1} A Y_k \,+\, A^* Y_{k-1}^n A (Y_k - Y_{k-1})\,. \end{aligned}$$

Using (4) we compute

$$\begin{aligned} \|Y_{k+1} - Y_k\| &= \left\| A^* \sum_{i=1}^{n} Y_k^{n-i}(Y_k - Y_{k-1}) Y_{k-1}^{i-1} A Y_k \,+\, A^* Y_{k-1}^n A (Y_k - Y_{k-1}) \right\| \\ &\leq (n+1)\|A\|^2(1+v)^n \|Y_k - Y_{k-1}\| \;\leq\; \cdots \\ &\leq \rho^k \|Y_1 - Y_0\| \;=\; \rho^k \|A\|^2, \end{aligned}$$

where $\rho = (n+1)\|A\|^2(1+v)^n < (n+1)\dfrac{n^n}{(n+1)^{n+1}}\left(1+\dfrac{1}{n}\right)^n = 1$. Hence the matrix sequence $\{Y_k\}$ is convergent. If $\bar{Y} = \lim\limits_{k\to\infty} Y_k$, then $\bar{Y}$ is a solution of the equation (2) and it satisfies $\|\bar{Y}\| \leq 1 + v$.

We will prove that $\bar{Y}$ is a positive definite matrix. Therefore we consider the following matrix equation

$$Y = I + \frac{1}{2}(A^*Y^nAY + YA^*Y^nA)\,. \tag{7}$$

It is easy to see that each solution Y of the equation (2) satisfies also (7). If Y is a solution of the equation (2), then we have $(I - A^*Y^nA)Y = I$. Hence $I - A^*Y^nA$ is an inverse matrix of Y. Therefore $Y(I - A^*Y^nA) = I$, and $Y = YA^*Y^nA + I$. Hence Y is a solution of the equation (7).

Now, we consider the matrix sequence $\{Z_k\}$:

$$Z_0 = I, \quad Z_{k+1} = I + \frac{1}{2}(A^*Z_k^nAZ_k + Z_kA^*Z_k^nA), \quad k = 0,1,2,\ldots. \tag{8}$$

It is obvious that Z_k is a Hermitian matrix for $k = 0,1,2,\ldots$. In a similar manner as the proof for Y_k, we can prove that $\|Z_k\| \leq 1+v$, $\|Z_{k+1} - Z_k\| \leq \rho^k\|A\|^2$ and there exists a Hermitian matrix Z, such that $Z = \lim_{k\to\infty} Z_k$, $\|Z\| \leq 1 + v$.

Let $M_k = \dfrac{1}{2}(A^*Z_k^nAZ_k + Z_kA^*Z_k^nA)$, then $M_k = M_k^*$. Let $\lambda(M_k)$ be an arbitrary eigenvalues of M_k. As

$$|\lambda(M_k)| \leq \|M_k\| \leq \|A\|^2\|Z_k\|^{n+1} < 1\,,$$

then $Z_{k+1} = I + M_k$ ($k = 0,1,2,\ldots$) is positive definite and Z is also a positive definite matrix.

We will prove that the equation (7) has unique solution Y, such that $\|Y\| < \dfrac{n+1}{n}$.

As for $\|Y_{k+1} - Y_k\|$, for $\|Z_{k+1} - Y\|$ we have

$$\begin{aligned}
\|Z_{k+1} - Y\| &= \left\| A^* \sum_{i=1}^{n} Z_k^{n-i}(Z_k - Y)Y^{i-1}AZ_k \;+\; A^*Y^nA(Z_k - Y) \right\| \\
&\le \|A\|^2 \left[\sum_{i=0}^{n} \|Z_k\|^{n-i}\|Y\|^i \right] \|Z_k - Y\| \\
&\le \|A\|^2 \left[\sum_{i=0}^{n} (1+v)^{n-i}\|Y\|^i \right] \|Z_k - Y\| \;\le\; \cdots \\
&\le \rho^k \|Z_1 - Y\|\,,
\end{aligned}$$

where $\rho = \|A\|^2 \sum\limits_{i=0}^{n} (1+v)^{n-i}\|Y\|^i < 1$. As $\bar{Y}$ is a solution of (2), (7) and $\|\bar{Y}\| \le 1 + v$ $< \dfrac{n+1}{n}$, then $\bar{Y} = Z$. The theorem is proved. $\square$

Let us note that (3) defines the iterative method for computing X_l^{-1} wich is linearly convergent. If the conditions of the theorem are fulfilled, then for every positive definite solution Y, different from the solution $\bar{Y}$ of (2) we have $\|Y\| \ge \dfrac{n+1}{n}$. Since $Y \ge \bar{Y}$, then $Y^{-1} \le \bar{Y}^{-1} = X_l$ and therefore if for the equation (1) there exists a maximal solution X_L then $X_L = X_l$.

3. Numerical experiments. We will consider the equation (1) when $n = 3$. We provide numerical experiments for computing a positive definite solution to that equation using MATLAB software and PENTIUM hardware. We use the suggested iterative method (3) and the following iterative method

$$X_{k+1} = I - A^* X_k^{-n} A, \;\; X_0 = \gamma\, I, \tag{9}$$

described by Hasanov and Ivanov [1]. The iterative methods (3) and (9) converge to the same positive definite solution of the matrix equation (1). We will use $\varepsilon = \|Z + A^T Z^{-3} A - I\|_\infty \le tol$ for $tol = 10^{-8}$ as a criteria to stop the iterative methods.

Example 1. We consider the matrix

$$A = \frac{1}{100} \begin{pmatrix} 16 & -9 & -8 \\ 11 & 16 & 5 \\ 4 & -8 & 18 \end{pmatrix}.$$

We compute the solution X of the equation (1) using the algorithm (9) and we start with various initial points, defined by the values of γ. With $\gamma = 1$ we need 8 iterations to compute X with accurate $\varepsilon = 7.54e-9$. For $\gamma = 0,955$ we need 7 iterations with accurate $\varepsilon = 5.10e-9$. In the case of $\gamma = 0,951$ this iterative method makes 7 iterations to reach an accurance of $\varepsilon = 5.83e-9$. For $\gamma = 0,75$ the number of iterations to reach an accurance of $\varepsilon = 1.54e-9$ is 10.

With this iterative method (3) for computing $X_l^{-1} = Y$ after 9 iterations we get accurance of $\varepsilon = \|Y^{-1} + A^*Y^3A - I\|_\infty = 9.42e-9 < 10^{-8}$.

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ВЪРХУ ЕДНО ПОЛОЖИТЕЛНО ОПРЕДЕЛЕНО РЕШЕНИЕ НА КЛАС ОТ НЕЛИНЕЙНИ МАТРИЧНИ УРАВНЕНИЯ

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Разгледано е матричното уравнение $X+A^*X^{-n}A = I$. Получен е нов итерационен метод за пресмятане на специално положително определено решение. Изпълнени са числени експерименти с различни методи за пресмятане на това решение и са сравнени резултатите от числените експерименти.