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SOLUTIONS AND PERTURBATION THEORY OF A SPECIAL MATRIX EQUATION II: PERTURBATION THEORY *

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Dedicated to Milko Petkov on the occasion of his 70th birthday

Perturbation theory of a special nonlinear matrix equation is discussed. New perturbation bounds for a special solution X_l of this equation are derived. The results are illustrated by numerical examples.

1. Introduction. We consider the nonlinear matrix equation

$$X + A^*X^{-n}A = Q\,, \tag{1}$$

where A, $Q \in \mathbb{C}^{m\times m}$ and Q is a positive definite matrix. The same equation is considered in [1, 2].

According to Theorem 3 [1] there exists a solution X_l of (1), such that

$$\frac{n}{(n+1)\|Q^{-1}\|}\, I \;<\; X_l \;\le\; Q \;-\; \frac{1}{\big(\|Q\|\,\|Q^{-1}\|\big)^{n-1}} A^*Q^{-n}A\,.$$

If the equation (1) has a maximal positive definite solution X_L, then $X_l \equiv X_L$. Ran and Reurings [3] have considered the more general nonlinear matrix equation $X+A^*\mathcal{F}(X)A = Q$ and perturbation bounds for its solutions. New perturbation bounds for the solution X_l of equation (1) are derived. Numerical experiments for computing perturbation bounds for X_l are executed. We use the notations introduced in [1].

2. Perturbation estimates. We consider the perturbed equation

$$\tilde{X} + \tilde{A}^*\tilde{X}^{-n}\tilde{A} = \tilde{Q}\,. \tag{2}$$

Theorem 1. *Let A, $\tilde{A}$, Q, $\tilde{Q} \in \mathbb{C}^{m\times m}$ with Q and $\tilde{Q}$ Hermitian positive definite. If*

$$\xi = \sqrt{\frac{n^n}{(n+1)^{n+1}}} - \|A\|\sqrt{\|Q^{-1}\|^{n+1}} > 0\,,$$

$$\|\tilde{A} - A\| < \frac{\sqrt{(n+1)^{n+1}} - \sqrt{n^n}}{\sqrt{\big[(n+1)\|Q^{-1}\|\big]^{n+1}}}\ \xi\,, \tag{3}$$

$$\|\tilde{Q} - Q\| \le \frac{1}{\|Q^{-1}\|}\ \left(1 - \sqrt[n+1]{(1-\xi)^2}\right)\,, \tag{4}$$

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then the solutions X_l *and* $\tilde{X}_l$ *of the matrix equations* (1) *and* (2) *exist and satisfy*

$$\|\tilde{X}_l - X_l\| \le \frac{1}{\varepsilon}\left[\|\Delta Q\| + \left(\frac{n+1}{n}\right)^n \|\tilde{Q}^{-1}\|^n \left(\|A\| + \|\tilde{A}\|\right)\|\Delta A\|\right]$$

and

$$\frac{\|\tilde{X}_l - X_l\|}{\|X_l\|} \le \frac{1}{\varepsilon}\left[\frac{\|\Delta Q\|}{\|Q\|}\,\frac{n+1}{n} + \frac{1}{n}\left(2 + \frac{\|\Delta A\|}{\|A\|}\right)\frac{\|\Delta A\|}{\|A\|}\right],$$

where $\varepsilon = 1 - \|A\|^2 \left(\frac{n+1}{n}\right)^{n+1} \sum_{i=1}^{n} \|\tilde{Q}^{-1}\|^i \, \|Q^{-1}\|^{n+1-i}$.

Proof. Let $\Delta Q = \tilde{Q} - Q$, $\Delta A = \tilde{A} - A$ and $\Delta X_l = \tilde{X}_l - X_l$. According to the identity $\tilde{Q}^{-1} = Q^{-1} - Q^{-1}\Delta Q\,\tilde{Q}^{-1}$, and using (4) we obtain the estimate

$$\|\tilde{Q}^{-1}\| \quad \le \quad \|Q^{-1}\| + \left(1 - \sqrt[n+1]{(1-\xi)^2}\right)\|\tilde{Q}^{-1}\|\,.$$

Then

$$\sqrt{\|\tilde{Q}^{-1}\|^{n+1}} \le \frac{\sqrt{\|Q^{-1}\|^{n+1}}}{1-\xi}\,. \tag{5}$$

Combining (3) and (5), we obtain

$$\begin{aligned}
\|\tilde{A}\|\sqrt{\|\tilde{Q}^{-1}\|^{n+1}} \quad &\le \quad \left(\|A\| + \|\Delta A\|\right)\sqrt{\|\tilde{Q}^{-1}\|^{n+1}} \\
&< \quad \left(\|A\| + \frac{\sqrt{(n+1)^{n+1}} - \sqrt{n^n}}{\sqrt{\left[(n+1)\|Q^{-1}\|\right]^{n+1}}}\,\xi\right)\frac{\sqrt{\|Q^{-1}\|^{n+1}}}{1-\xi} \\
&= \quad \frac{\|A\|\sqrt{\left[(n+1)\|Q^{-1}\|\right]^{n+1}} + \left(\sqrt{(n+1)^{n+1}} - \sqrt{n^n}\right)\xi}{(1-\xi)\sqrt{(n+1)^{n+1}}} \\
&= \quad \sqrt{\frac{n^n}{(n+1)^{n+1}}}\,.
\end{aligned}$$

According to Theorem 3 [1] it follows the existence of the solutions X_l and $\tilde{X}_l$ of the equations (1) and (2). From Corollary 4 and Corollary 5 [1] it follows that these solutions satisfy the inequalities :

$$\|Q^{-1}\| \;\le\; \|X_l^{-1}\| \;<\; \frac{n+1}{n}\|Q^{-1}\|\,; \qquad \frac{n}{n+1}\|Q\| \;<\; \|X_l\| \;\le\; \|Q\|\,; \tag{6}$$

$$\|\tilde{Q}^{-1}\| \;\le\; \|\tilde{X}_l^{-1}\| \;<\; \frac{n+1}{n}\|\tilde{Q}^{-1}\|\,; \qquad \frac{n}{n+1}\|\tilde{Q}\| \;<\; \|\tilde{X}_l\| \;\le\; \|\tilde{Q}\|\,. \tag{7}$$

Note that

$$\|A\|\sqrt{\|\tilde{Q}^{-1}\|^{n+1}} \le (\,\|A\| + \|\Delta A\|\,)\sqrt{\|\tilde{Q}^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}\,.$$

Consider the identity

$$\Delta Q = \Delta X_l + A^*\tilde{X}_l^{-n}\Delta A + \Delta A^*\tilde{X}_l^{-n}\tilde{A} - A^*\sum_{i=1}^{n}\tilde{X}_l^{-i}\Delta X_l\,X_l^{i-(n+1)}\,A \tag{8}$$

Using inequalities (6) and (7) we obtain

$$\begin{aligned}\kappa &\equiv \left\| \Delta X_l - A^* \sum_{i=1}^{n} \tilde{X}_l^{-i} \Delta X_l \, X_l^{i-(n+1)} \, A \right\| \\ &\geq \|\Delta X_l\| - \|A\|^2 \, \|\Delta X_l\| \sum_{i=1}^{n} \|\tilde{X}_l^{-i}\| \, \|X_l^{i-(n+1)}\| \;\geq\; \delta \, \|\Delta X_l\| \, ,\end{aligned}$$

where $\delta = 1 - \|A\|^2 \left(\frac{n+1}{n}\right)^{n+1} \sum_{i=1}^{n} \|\tilde{Q}^{-1}\|^i \, \|Q^{-1}\|^{n+1-i} > 0$.
Hence, from identity (8) we have

$$\begin{aligned}\varepsilon \, \|\Delta X_l\| &\leq \kappa = \left\| \Delta Q + A^* \tilde{X}^{-n} \Delta A + \Delta A^* \tilde{X}^{-n} \tilde{A} \right\| \\ &\leq \|\Delta Q\| + \|A\| \, \|\tilde{X}^{-n}\| \, \|\Delta A\| + \|\Delta A\| \, \|\tilde{X}^{-n}\| \, \|\tilde{A}\| \\ &\leq \|\Delta Q\| + \left(\frac{n+1}{n}\right)^n \|\tilde{Q}^{-1}\|^n \left(\|A\| + \|\tilde{A}\| \right) \|\Delta A\| \, .\end{aligned}$$

Hence

$$\|\Delta X_l\| \leq \frac{1}{\varepsilon} \left[\|\Delta Q\| + \left(\frac{n+1}{n}\right)^n \|\tilde{Q}^{-1}\|^n \left(\|A\| + \|\tilde{A}\| \right) \|\Delta A\| \right] .$$

Note that

$$\|X_l\| \geq \frac{n}{n+1} \, \|Q\| \geq \frac{n}{n+1} \, \|Q^{-1}\|^{-1} \, .$$

Combining the last estimates, we obtain

$$\begin{aligned}\frac{\|\Delta X_l\|}{\|X_l\|} &\leq \frac{1}{\varepsilon} \left[\frac{\|\Delta Q\|}{\|X_l\|} + \left(\frac{n+1}{n}\right)^n \|\tilde{Q}^{-1}\|^n \left(\|A\| + \|\tilde{A}\| \right) \frac{\|\Delta A\|}{\|X_l\|} \right] \\ &= \frac{1}{\varepsilon} \left[\frac{\|\Delta Q\|}{\|Q\|} \, \frac{\|Q\|}{\|X_l\|} + \left(\frac{n+1}{n}\right)^n \|\tilde{Q}^{-1}\|^n \left(\|A\| + \|\tilde{A}\| \right) \frac{\|\Delta A\|}{\|A\|} \, \frac{\|A\|}{\|X_l\|} \right] \\ Q^{-1}\|^n \|Q^{-1}\| &\leq \frac{1}{\varepsilon} \left[\frac{\|\Delta Q\|}{\|Q\|} \, \frac{n+1}{n} + \frac{1}{n} \left(2 + \frac{\|\Delta A\|}{\|A\|} \right) \frac{\|\Delta A\|}{\|A\|} \right] .\end{aligned}$$

The theorem is proved. □

Theorem 2. *Let* $\|A\| \sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\frac{n^n}{(n+1)^{n+1}}}$ *and* $\tilde{X}$ *be a positive definite matrix, which approximates the solution* X_l *of the matrix equation* (1). *If*

$$\nu = 1 - \|A\| \sum_{i=1}^{n} \|\tilde{X}^{-i} A\| \left(\frac{n+1}{n} \, \|Q^{-1}\| \right)^{n+1-i} > 0 \, ,$$

then

$$\|\Delta X\| \leq \frac{1}{\nu} \|R(\tilde{X})\|$$

and

$$\frac{\|\Delta X\|}{\|X\|} \leq \frac{n+1}{n \, \nu} \, \frac{\|R(\tilde{X})\|}{\|Q\|} \, , \tag{9}$$

where $R(\tilde{X}) = \tilde{X} + A^* \tilde{X}^{-n} A - Q$.

Proof. Note that $\tilde{X}$ is a solution of the matrix equation $X + A^* X^{-n} A = \tilde{Q}$, where

$\tilde{Q} = Q + R(\tilde{X})$. Then the identity (8) can be written as follows

$$R(\tilde{X}) = \tilde{X} - X_l - A^* \sum_{i=1}^{n} \tilde{X}^{-i}(\tilde{X} - X_l) X_l^{i-(n+1)} A \,.$$

We obtain

$$\begin{aligned}
\|R(\tilde{X})\| &\geq \|\tilde{X} - X_l\| - \|\tilde{X} - X_l\| \sum_{i=1}^{n} \|\tilde{X}^{-i} A\| \|X^{i-(n+1)} A\| \\
&\geq \|\tilde{X} - X_l\| - \|A\| \, \|\tilde{X} - X_l\| \sum_{i=1}^{n} \|\tilde{X}^{-i} A\| \, \|X_l^{i-(n+1)}\| \\
&\geq \|\tilde{X} - X_l\| \left(1 - \|A\| \sum_{i=1}^{n} \|\tilde{X}^{-i} A\| \left(\frac{n+1}{n} \|Q^{-1}\| \right)^{n+1-i} \right) \\
&= \nu \, \|\tilde{X} - X_l\| \,.
\end{aligned}$$

Since $\dfrac{\|Q\|}{\|X_l\|} \leq \dfrac{n+1}{n}$ we have

$$\frac{\|\tilde{X} - X_l\|}{\|X_l\|} \leq \frac{1}{\nu} \frac{\|R(\tilde{X})\|}{\|Q\|} \frac{\|Q\|}{\|X_l\|} \leq \frac{n+1}{n\,\nu} \frac{\|R(\tilde{X})\|}{\|Q\|} \equiv res_\nu \,. \qquad \square$$

Corollary 3. *Let* $\|A\| \sqrt{\|Q^{-1}\|^{n+1}} < \sqrt{\dfrac{n^n}{(n+1)^{n+1}}}$ *and* $\tilde{X}$ *is a positive definite matrix, which approximates the solution* X_l *of the matrix equation (1).*
If $\|\tilde{X}^{-1}\| \leq \dfrac{n+1}{n} \|Q^{-1}\|$, *then*

$$\|\tilde{X} - X_l\| \leq \frac{1}{\nu_1} \|R(\tilde{X})\| \quad \textit{and} \quad \frac{\|\tilde{X} - X_l\|}{\|X_l\|} \leq \frac{n+1}{n\,\nu_1} \frac{\|R(\tilde{X})\|}{\|Q\|} \,,$$

where $\nu_1 = 1 - \dfrac{(n+1)^{n+1}}{n^n} \|A\|^2 \|Q^{-1}\|^{n+1}$ *and* $R(\tilde{X}) = \tilde{X} + A^* \tilde{X}^{-n} A - Q$.

Example. Consider the equation (1) with coefficient matrices

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and the solution } X = \begin{pmatrix} 1 & 0 \\ 0 & 1 - a^2 \end{pmatrix},$$

where $0 < a < 1$. For the perturbed equation $\tilde{X} + A^T \tilde{X}^{-2} A = \tilde{Q}$ we have

$$\tilde{X} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 - a^2 \end{pmatrix} \text{and } \tilde{Q} = \tilde{X} + A^T (\tilde{X})^{-2} A, \text{ where } \epsilon = 10^{-2j}, \; j = 2, 3, 4.$$

Some numerical results on the relative perturbation bounds (9) and Ran and Reurings [3] perturbation bound are shown in Table 1.

Table 1. $a = \frac{1}{3} - 10^{-j}$.

j	Real error	(9)	Ran & Reurings [3]
	$\frac{\|\Delta X\|}{\|X\|}$	res_ν	Prop. 4.1 $M_{S(m)} = \frac{27}{4} \|Q^{-1}\|^3$
2	$1.0000e-04$	$2.4673e-04$	$1.9499e-003$
3	$1.0000e-06$	$2.5605e-06$	$1.9270e-004$
4	$1.0000e-08$	$2.5703e-08$	$1.9248e-005$

The results show that our perturbation estimate (9) is better than the Ran and Reurings estimate for the considered example.

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РЕШЕНИЯ И ПЕРТУРБАЦИОННА ТЕОРИЯ ЗА СПЕЦИАЛНО МАТРИЧНО УРАВНЕНИЕ II: ПЕРТУРБАЦИОННА ТЕОРИЯ

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Разгледано е пертурбационна теория на специално нелинейно матрично уравнение. Дадени са пертурбационни оценки за специално решение X_l на разгледаното уравнение. Резултатите са илюстрирани с числени примери.