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SOME RESULTS ON BOUNDED AND INFINITELY DIVISIBLE RANDOM VECTORS

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In this paper we consider bounded from below infinitely divisible random vectors and give similar statements about bounded from above ones. We investigate the relationship between the infinimums of supports of the Lévy measure and distribution function of such a vector. The results generalize those of Baxter & Shappiro (1960), which concern the one-dimensional case.

1. Introduction. By definition a d-dimensional random vector X, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is infinitely divisible (ID) if for all $n \in \mathcal{N}$ there exist independent identically distributed (i.i.d.) random vectors (r.v's) $X_{1n}, X_{2n}, \dots, X_{nn}$ such that $X \stackrel{\mathrm{d}}{=} X_{1n} + X_{2n} + \dots + X_{nn}$. Here and further on we use the notation $\stackrel{\mathrm{d}}{=}$ in the sense of "coincide in distribution". The distribution of such a vector is uniquely determined by its characteristic function (ch.fct.) $\varphi(z)$. In general $\ln\varphi(z)$ has the form

$$(1)\qquad i(\gamma, z) - \frac{Q(z)}{2} + \int\limits_{\mathbf{R}^d\setminus\{\mathbf{0}\}} (e^{i(z,x)} - 1 - \frac{i(z,x)}{1+(x,x)})\Pi(dx),$$

where $z \in \mathbf{R}^d$, γ is a constant vector in $\mathbf{R}^d$, Π is the Lévy measure of X, $\Pi(\{|x|>1\}) < \infty$,

$$\lim_{\varepsilon\downarrow 0} \int\limits_{\varepsilon<|x|<1} (x,x)\Pi(dx) < \infty$$

and

$$Q(z) = \lim_{\varepsilon\downarrow 0} \int\limits_{|x|<\varepsilon} \frac{(z,x)^2}{1+(x,x)}\Pi(dx) < \infty.$$

As it is known, $\ln\varphi(z)$ can be also written in the equivalent form

$$(2)\qquad i(a(\tau), z) - \frac{Q(z)}{2} + \int\limits_{\mathbf{R}^d\setminus\{\mathbf{0}\}} (e^{i(z,x)} - 1 - i(z,x)I\{0 < |x| < \tau\})\Pi(dx)$$

where $\tau\mathbf{e}$ is a continuity point of $\Pi(x)$, e $\in \mathbf{R}^d$, e $= (1,\dots,1)$ and

$$(a(\tau), z) = (\gamma, z) + \int\limits_{0<|x|<\tau} (z,x)\Pi(dx) - \int\limits_{0<|x|} \frac{(z,x)}{1+(x,x)}\Pi(dx).$$

Note: With an abuse of notation we denote by the same letter Π the Lévy measure and the corresponding distribution function. More precisely, $\Pi(A_x^c) = -\Pi(x)$. Here and further on $A_x^c = [0,\infty)^d \backslash [0,x)$.

The properties of multidimensional ID r.v's seem to be investigated at first by Rvacheva (1962). It is well known that if an ID r.v. is almost surely (a.s.) bounded, then $X \overset{\text{a.s.}}{=}$ constant. Further, in this paper we consider the bounded from below ID r.v's. The statements concerning bounded from above r.v's are analogous. Our results generalize those of Baxter & Shappiro (1960) which consider the one-dimensional case.

Recall that the d-dimensional r.v. X is bounded from below if there exists a vector a$\in \mathbf{R}^d$ such that $\mathbf{P}(X > a) = 1$, where $\{X > a\} = \{X_1 > a_1, \ldots, X_d > a_d\}$.

Our main results follow below.

Proposition 1. *A necessary and sufficient condition for the ID r.v. X to be bounded from below is its Lévy measure Π and $Q(z)$ to satisfy the following three conditions:*

1) $\Pi(\{y \in \mathbf{R}^d : \bigcup_{i=1}^{d} y_i < 0\}) = 0$;

2) $Q(z) = 0$ *for all* $z > 0$;

3) $\lim_{\varepsilon \downarrow 0} \int_{[\varepsilon,1)^d \backslash \{e\varepsilon\}} (x,z)\Pi(dx) < \infty$, *where* $e \in \mathbf{R}^d$, $e = (1,\ldots,1)$.

Note that df Π is an increasing and non-positive one.

Proposition 2. *Let X be ID with df F and Lévy measure Π and $Q(z)$ satisfying conditions* 1), 2) *and* 3). *Then*

i) there exists $a_0 \in \mathbf{R}^d$ *such that* $\mathbf{P}(X > a_0) = 1$ *and*

$$(a_0, z) = (\gamma, z) - \int_{[0,\infty)^d \backslash \{\mathbf{0}\}} \frac{(z,x)}{1+(x,x)} \Pi(dx) = (a(\tau), z) - \int_{[0,\tau)^d \backslash \{\mathbf{0}\}} (z,x)\Pi(dx);$$

ii) $a_0 = \inf\ \mathrm{Supp}\ F - \inf\ \mathrm{Supp}\ \Pi$;

iii) the ch.fct. of X has the form

$$\varphi(z) = \exp\{i(a_0,z) + \int_{[0,\infty)^d \backslash \{\mathbf{0}\}} (e^{i(z,x)} - 1)\Pi(dx)\}. \tag{3}$$

Proposition 3. *A necessary and sufficient condition for the ID r.v. X to be bounded from above is its Lévy measure Π and $Q(z)$ to satisfy the following three conditions:*

1′) $\Pi(\{y \in \mathbf{R}^d : \bigcup_{i=1}^{d} y_i > 0\}) = 0$;

2′) $Q(z) = 0$ *for all* $z > 0$;

3′) $-\lim_{\varepsilon \downarrow 0} \int_{[-1,-\varepsilon)^d \backslash \{-e\varepsilon\}} (x,z)\Pi(dx) < \infty$, *where* $e \in \mathbf{R}^d$, $e = (1,\ldots,1)$.

Here *df* Π is an increasing and non-negative.

Proposition 4. *Let X be ID with df F and Lévy measure Π and $Q(z)$ satisfying conditions* 1′), 2′) *and* 3′). *Then* *i′) there exists* $b_0 \in \mathbf{R}^d$ *such that* $\mathbf{P}(X < b_0) = 1$ *and*

$$(b_0, z) = (\gamma, z) - \int\limits_{(-\infty,0]^d\setminus\{\mathbf{0}\}} \frac{(z,x)}{1+(x,x)}\Pi(dx) = (a(\tau), z) - \int\limits_{(-\tau,0]^d\setminus\{\mathbf{0}\}} (z,x)\Pi(dx);$$

ii′) $b_0 = \sup\ \mathrm{Supp}\ F - \sup\ \mathrm{Supp}\ \Pi$*;*
iii′) the ch.fct. of X *has the form*

$$\varphi(z) = \exp\{i(b_0, z) + \int\limits_{(-\infty,0]^d\setminus\{\mathbf{0}\}} (e^{i(z,x)} - 1)\Pi(dx)\}. \tag{4}$$

2. Proofs. In order to prove Proposition 1 we need the following three lemmas.

Lemma 1. *Let* X *be bounded from below ID r.v. Then*

1) $\Pi(\{y \in \mathbf{R}^d : \bigcup_{i=1}^d y_i < 0\}) = 0.$

Proof. X is bounded from bellow, i.e. there exists $a_0 \in \mathbf{R}^d$ such that $\mathbf{P}(X - a_0 \geq 0) = 1$. So, $X - a_0$ is ID, too. Then, for all n$\in \mathcal{N}$, there exist i.i.d. r.v.'s $X_{n1}, \dots, X_{nn}$ with fd's F_n such that $X - a_0 = X_{n1} + \dots + X_{nn}$. So, $\mathbf{P}(X_{n1} + \dots + X_{nn} \geq \mathbf{0}) = 1$ and we get $\mathbf{P}(\bigcup_{i=1}^d \{X_{n1}^i < 0\}) = 0$. Let $S_r := \{y \in \mathbf{R}^d : |y| > r\}$, ${S_r}^+ := \{y \in \mathbf{R}^d : |y| > r, y \geq \mathbf{0}\}$ and assume that 1) is not valid. Then, there exists $r_0 > 0$ such that $\Pi_{X-a_0}(S_{r_0}) > \Pi_{X-a_0}(S_{r_0}^+)$. By the multidimensional Central Criterion of Convergence (CCC) (cf. Rvacheva, 1962, Th.2.3), $n\mathbf{P}(X_{n1} \in S_{r_0}^+) \to \Pi_{X-a_0}(S_{r_0}^+)$ and $n\mathbf{P}(X_{n1} \in S_{r_0}) \to \Pi_{X-a_0}(S_{r_0})$. On the other hand, $\mathbf{P}(X_{n1} \in S_{r_0}) =$

$$= \mathbf{P}(X_{n1} \in S_{r_0}, \bigcup_{i=1}^d \{X_{n1}^i < 0\}) + \mathbf{P}(X_{n1} \in S_{r_0}, X_{n1} \geq 0) = \mathbf{P}(X_{n1} \in S_{r_0}^+).$$

So we get a contradiction and thus the proof is complete since $X \sim \mathrm{ID}(\gamma + a_0, \Pi_{X-a_0})$.

Lemma 2. *Let* X *be bounded from below ID r.v. Then*

2) $Q(t) = 0$*, for all* $t \in \mathbf{R}^d$*.*

Sketch of the proof. Assume that $Q(t)$ is not identically zero. Then X has a normal component, and consequently, is unbounded.

Lemma 3. *Let* X *be bounded from below ID r.v. Then*

3) $\lim\limits_{\varepsilon \downarrow 0} \int\limits_{[\varepsilon,1)^d \setminus \{e\varepsilon\}} (x, z)\Pi(dx) < \infty.$

Proof. Let $X - a_0 \overset{\mathrm{d}}{=} X_{n1} + \dots + X_{nn}$, where X_{ni} are i.i.d. for $i = 1, \dots, n$ and have df F_n on $[0, \infty)^d$ (cf. the proof of Lemma 1.) Let $\mathbf{e}\tau$ be a point of continuity of Π. Consequently,

$$\int\limits_{[-\mathbf{e}\tau, \mathbf{e}\tau]} (x, z)dF_n(x) = \int\limits_{[0, \mathbf{e}\tau]} (x, z)dF_n(x) \text{ for all } \tau \in (0, \infty).$$

On the other hand, by CCC:

1.) $\lim\limits_{n\to\infty} n \int\limits_{[0,\mathbf{e}\tau]} (x, z)dF_n(x) < \infty;$

2.) $\lim\limits_{n\to\infty} nF_n(M) = \Pi(M) < \infty$, for all $M \subset \mathbf{R}^d$ $0 \notin M$.

So, by integration by parts, we get:

$$\begin{aligned}
\infty &> \lim_{n\to\infty} n \int\limits_{[0,\mathbf{e}\tau]} (x,z) dF_n(x) = \\
&= -\lim_{n\to\infty} \sum_{i=1}^{d} t_i \int\limits_0^\tau x_i dn\{F_n(\tau,\ldots,\tau) - F_n(\tau,\ldots,\tau,x_i,\tau,\ldots,\tau)\} = \\
&= \lim_{n\to\infty} \sum_{i=1}^{d} t_i \int\limits_0^\tau n\{F_n(\tau,\ldots,\tau) - F_n(\tau,\ldots,\tau,x_i,\tau,\ldots,\tau)\} dx_i = \\
&\geq \lim_{n\to\infty} \sum_{i=1}^{d} t_i \int\limits_\varepsilon^\tau n\{F_n(\tau,\ldots,\tau) - F_n(\tau,\ldots,\tau,x_i,\tau,\ldots,\tau)\} dx_i = \\
&= \sum_{i=1}^{d} t_i \int\limits_\varepsilon^\tau \Pi(A_{(\tau,x_i)}) dx_i,
\end{aligned}$$

where $A_{(\tau,x_i)} = \{[0,\tau) \times \cdots \times [0,\tau) \times (x_i,\tau) \times [0,\tau) \times \cdots \times [0,\tau)\}$,

By the Monotone Convergence Theorem $\sum\limits_{i=1}^{d} t_i \int\limits_0^\tau \Pi(A_{(\triangle au,x_i)}) dx_i < \infty$. Integrating again by parts, we complete the proof.

Proof of Proposition 1. The necessity follows by Lemma 1, Lemma 2 and Lemma 3. Sufficiency follows by i) in Proposition 2.

Proof of Proposition 2. First, we will show i). Conditions ii) and iii) will be its consequences.

By condition 3), we have
$\int\limits_{[0,1)^d\setminus\{\mathbf{0}\}} \frac{(x,z)}{1+(x,x)} \Pi(dx) < \infty$. Let $\lambda_i := \int\limits_{[0,1)^d\setminus\{\mathbf{0}\}} \frac{x_i}{1+(x,x)} \Pi(dx)$, and let $a_0 = \gamma - \lambda$.
Then, $\varphi(z) = \exp\{i(a_0,z) + \int\limits_{[0,\infty)^d\setminus\{\mathbf{0}\}} (e^{i(z,x)} - 1)\Pi(dx)\}$ and $X - a_0$ has the characteristic function

$$Ee^{(z,X-\gamma+\lambda)} = Ee^{(z,X-a_0)} = \exp\left\{ \int\limits_{[0,\infty)^d\setminus\{\mathbf{0}\}} (e^{i(z,x)} - 1)\Pi(dx) \right\} \tag{5}$$

We consider two cases.

Case 1: $-\Pi(0+) := \lim\limits_{\varepsilon\downarrow 0} \Pi(A^c_{e\varepsilon}) < \infty$. It is not difficult to see that $a_0 - X$ has *df* $F(z) = e^{\Pi(0+)} \sum\limits_{k=0}^{\infty} \frac{(-\Pi(-z))^{*k}}{k!}$ for all $z \in (-\infty, 0]^d \setminus \{\mathbf{0}\}$ and $F(z) = 1$, elsewhere. Then, $F(0) = \mathbf{P}(a_0 - X \leq 0) = \mathbf{P}(a_0 \leq X) = 1$, i.e. $X \geq a_0$ a.s.

To prove ii) note that

$$\begin{aligned}
&\inf\ \mathrm{Supp}\ \{(X-\gamma+\lambda)\} = -\sup\{y\in(-\infty,0]^d : F(y)<1\} = \\
&= -\sup\left\{y\in(-\infty,0]^d : \sum_{k=0}^{\infty}\frac{(-\Pi(-y))^{*k}}{k!} < e^{-\Pi(0+)}\right\} = \\
&= -\sup\left\{y\in(-\infty,0]^d : \sum_{k=0}^{\infty}\left(\frac{-\Pi(-y)}{-\Pi(0+)}\right)^{*k}\frac{(-\Pi(0+))^k}{k!} < e^{-\Pi(0+)}\right\} = \\
&= -\sup\{y\in(-\infty,0]^d : -\Pi(-y) < -\Pi(0+)\} = \\
&= \inf\{y\in[0,\infty)^d : \Pi(y) > \Pi(0+)\} = \inf\ \mathrm{Supp}\ \Pi.
\end{aligned}$$

Consequently $a_0 = \inf\ \mathrm{Supp}\ X - \inf\ \mathrm{Supp}\ \Pi$.

iii) is obvious.

Case 2: Let $\lim\limits_{\varepsilon\downarrow 0}\Pi(A^c_{e\varepsilon}) = \inf ty$. We choose a sequence $\varepsilon_n \downarrow 0$, as $n\to\infty$, such that $\varepsilon_n\mathbf{e}$ are continuity points of $\Pi(x)$ for all n$\in\mathcal{N}$. Let X_n be a sequence of independent ID r.v's with df F_n(x) and characteristic function

$$i(\gamma,z) + \int\limits_{\mathbf{R}^d\setminus\{\mathbf{0}\}} (e^{i(z,x)} - 1 - \frac{i(z,x)}{1+(x,x)})\Pi_n(dx),$$

whose Lévy measure Π_n has the form

$$\Pi_n(A^c_x) = \begin{cases} \Pi(A^c_x) & x\in A^c_{\varepsilon_n} \\ \Pi(A^c_{\varepsilon_n}) & x\in[0,\epsilon_n)^d \\ 0 & \text{elsewhere} \end{cases}. \tag{6}$$

So, Π_n satisfies the finiteness condition of the previous case for all $n\in\mathcal{N}$. Then,

$$X^i_n \overset{a.s.}{>} \gamma_i - \varepsilon_n - \int\limits_{[\varepsilon_n,1)^d\setminus\{\varepsilon_n\mathbf{e}\}} \frac{x_i}{1+(x,x)}\Pi_n(dx) \tag{7}$$

Since $\Pi_n(u) \underset{n\to\infty}{\to} \Pi(u)$ weakly and $\lim\limits_{n\to\infty}\int\limits_{0<|x|<\varepsilon_n}(z,x)^2\Pi_n(dx) = 0$ by CCC we get $X_n \overset{d}{\underset{n\to\infty}{\to}} X$. Now, taking the limit as $n\to\infty$ in (7), we finally get

$$X^i \overset{a.s.}{>} \gamma_i - \int\limits_{[0,1)^d\setminus\{\mathbf{0}\}} \frac{x_i}{1+(x,x)}\Pi(dx).$$

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НЯКОИ РЕЗУЛТАТИ ЗА ОГРАНИЧЕНИТЕ И БЕЗГРАНИЧНО ДЕЛИМИ СЛУЧАЙНИ ВЕКТОРИ

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В тази статия разглеждаме ограничените отдолу и формулираме аналогични твърдения за ограничените отгоре базгранично делими случайни вектори. Изследваме връзката между долните граници на носителите на мярката на Леви и функцията на разпределение на такъв вектор.
Статията обобщава резултатите на Бакстер и Шапиро (1960), където е разгледан едномерния случай.