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ON PROPERTIES OF PERTURBATION BOUNDS*

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In this paper we introduce and study properties of perturbation bounds such as asymptotical sharpness, asymptotical exactness and exactness. These properties are essential in the perturbation analysis of algebraic matrix equations in linear algebra and control theory.

1. Introduction. In this paper we discuss the main properties of perturbation bounds that are used in the analysis of computational problems with either explicit or implicit solution. Some important concepts in this area are introduced and illustrated by examples. We consider matrix explicit $X = \Phi(A)$ or implicit $F(A, X) = 0$ problems, where X is the matrix solution and A is a collection of data matrices. Typical problems here are the computation of the matrix exponential $X = \exp(A)$ and the solution of the matrix quadratic equation $A_1 + A_2X + XA_3 + XA_4X = 0$, $A = (A_1, \ldots, A_4)$. In what follows $\|\cdot\|$ is a matrix norm, $\|A\|_g = [\|A_1\|, \ldots, \|A_r\|]^\top \in \mathbf{R}^r_+$ is the generalized norm of the matrix r-tuple $A = (A_1, \ldots, A_r)$ and $\preceq$ is the component-wise partial order relation on $\mathbf{R}^r$.

2. Definitions and properties. The literature in perturbation theory is rich in various types of perturbation bounds [3, 2, 1]. However, for many of them no quantitative or qualitative measures of exactness are discussed. Also, often the domains of applicability of some bounds are not known (or not stated) clearly. This is particularly true for local perturbation bounds, based on condition numbers.

We believe that a perturbation bound should obey some natural requirements: (i) the bound should be *rigourous* in the sense that its domain of applicability should be a priori known and clearly displayed. In its domain of applicability the bound should give true estimates for the perturbed quantities; (ii) the bound should be *sharp* or *exact* in some sense, which also must be clear to the potential user; (iii) if the bound is too pessimistic for some cases, this should be clearly stated.

The list may continue, e.g., we may require that the bound is *general* in the sense that it imposes minimum restrictions and is thus applicable to a general class of problems.

These requirements do not mean that bounds with unknown domain of applicability, as well as some heuristic (or experimentally stated) bounds are practically useless. Such bounds are of practical use, but one should be aware of the fact that the bound is not proven to be rigorous.

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Perturbation and error estimates should be included in software tools for solving engineering and scientific problems. Without such estimates the corresponding software can not be recognised as reliable.

In what follows we present the concepts of sharpness, exactness and attainability of perturbation bounds.

Let X be the matrix solution of a regular computational problem with matrix data $A = (A_1, \ldots, A_r)$, and $X + \delta X$ be a solution, corresponding to the perturbed data $A+\delta A$. In case of an explicit problem we have $X = \Phi(A)$, where the Φ function is locally Lipschitz, and $\delta X = \Psi(A, \delta A) := \Phi(A + \delta A) - \Phi(A)$. In case of an implicit problem, let X be the solution of the equation $F(A, X) = 0$. Here Φ satisfies $F(B, \Phi(B)) = 0$ for all B from a neighbourhood of the nominal data A. We set $\delta_X := \|\delta X\| = \|\Psi(A, \delta A)\|$, $\delta_X = \delta_X(\delta A)$, denoting the dependence of δ_X only on δA for a fixed A.

Suppose that we have a perturbation bound

$$\delta_X \leq f(\|\delta A\|_g),\ \delta A \in \mathcal{D}, \tag{1}$$

where the domain $\mathcal{D} \subset \mathbf{R}^r_+$ contains a set $\{z \in \mathbf{R}^r : 0 \leq z_i \leq \rho_i\}$ of positive measure ($\rho_i > 0$ for all $i = 1, \ldots, r$). Let also $\omega(\delta) := \max\{\delta_X(\delta A) : \|\delta A\|_g \preceq \delta\}$.

Definition 1. *The perturbation $\delta A = (\delta A_1, \ldots, \delta A_r)$ is full if all δA_i are non-zero. The bound* (1) *is* asymptotically sharp *if there exists E such that $\delta_X(\varepsilon E) = f(\varepsilon\|E\|_g)$ $+o(\varepsilon)$, $\varepsilon \to 0$.*

Thus, the asymptotical sharpness is a property, connected with the existence of an one-parametric family of full perturbations $\{\varepsilon E\}$, $\varepsilon \to 0$, for which the bound (1) is asymptotically equivalent to the maximum possible perturbation in the solution. More precisely, an asymptotically sharp bound is asymptotically equivalent to the actual perturbation for the given family of full perturbations in the sense that $\lim_{\varepsilon\to+0} f(\varepsilon\|E\|_g)/\delta_X(\varepsilon E) = 1$.

A perturbation bound *should be* asymptotically sharp, otherwise it may be substantially improved. Unfortunately, many bounds that are used in the literature, are not asymptotically sharp. Moreover, some wide spread bounds are not even bounds in the strict sense, since they may underestimate the actual perturbation in the solution nevetheless how small the perturbation in the data is. Consider for example the scalar problem $x = \varphi(a)$ with φ differentiable at a. The chopped condition number based bound is $|\delta x| \leq |\varphi'(a)|\,|\delta a|$. For $\varphi(a) = a^2$ and $a = 0$ this bound reduces to $\delta x = 0$ which is not true for all $\delta a \neq 0$.

Definition 2. *The bound* (1) *is* asymptotically exact *if $\omega(\delta) = f(\delta) + o(\|\delta\|)$, $\delta \to 0$, and* exact *if $\mathcal{D} = \Omega$ and $f = \omega$.*

Exact bounds are available only in rare cases. For example, given the scalar problem $x = a^2$, the exact bound is $f(\delta) = \delta(2|a| + \delta)$.

If a bound is not exact or even asymptotically exact, then one would like to estimate quantitatively how far is this bound from the set of asymptotically exact bounds.

Definition 3. *The quantity* $\mathrm{mes}(f) := \lim_{\alpha\to 0} \sup\{f(\delta)/\omega(\delta) : \|\delta\| \leq \alpha\}$ *is the* measure of asymptotic exactness *of the bound* (1).

If it is possible to find the true value of $\mathrm{mes}(f)$ for a bound, which is only asymptotically sharp but not asymptotically exact, then for each $\tau > 1$, close to 1, we can define a new local bound $\delta_X \leq \tau f(\delta)/\mathrm{mes}(f) + o(\|\delta\|)$, $\delta \to 0$, which is asymptotically close to $\omega(\delta)$.

Some perturbation bounds known in the literature have a property which may be defined as follows. Denote by $\mathcal{D}_+ \subset \mathcal{D}$ the set of all $\delta \in \mathcal{D}$ with $\delta \succ 0$.

Definition 4. *The bound* (1) *is* attainable *if there exists a manifold* $\mathcal{M} \subset \mathcal{D}_+$ *of dimension* $\dim(\mathcal{M}) = r - 1$*, such that* $f(\delta) = \omega(\delta)$ *for* $\delta \in \mathcal{M}$.

Definition 5. *The bound* (1) *is* almost achievable *if for any positive* $\tau < 1$ *there exists a pertutbation* δA *such that* $\|\delta X\| = \tau f(\|\delta A\|_g)$.

Often, attainable bounds are not even asymptotically sharp. In turn, an asymptotically exact bound may not be attainable.

The next two examples of scalar linear equations illustrate the above concepts.

Example 1. Consider the scalar equation $ax = c$, $a \neq 0$, with solution $x = c/a$, and let $\delta_c := |\delta c|$, $\delta_a := |\delta a|$ and $\delta_x := |\delta x|$ be the absolute perturbations in c, a and x. For $\delta a \neq -a$ we have $\delta x = (\delta c - x\delta a)/(a + \delta a)$. Hence $\omega(\delta) = (\delta_c + |x|\delta_a)/(|a| - \delta_a)$ and the domain Ω for $\delta = [\delta_c, \delta_a]^\top$ is $\mathbf{R}_+ \times [0, |a|)$. Consider the following expression in δ, depending on two parameters $\alpha \geq 1$ and $\beta \geq 0$, $f_{\alpha,\beta}(\delta) := (\alpha(\delta_c + |x|\delta_a))/(|a| - \beta\delta_a)$.

We have the following five possible cases.

1. If $\alpha = 1$ and $\beta < 1$, then the inequality $\delta_x \leq f_{1,\beta}(\delta)$ may not be true and hence $f_{1,\beta}(\delta)$ is not a bound in the strict sense.

2. If $\alpha = \beta = 1$, then the bound is exact, and hence asymptotically sharp, asymptotically exact and attainable.

3. If $\alpha = 1$ and $\beta > 1$, then the bound is asymptotically exact and hence asymptotically sharp, but not exact and not attainable. Here $\mathcal{D} = \mathcal{R}_+ \times [0, |a|/\beta)$ is a proper subset of Ω.

4. If $\alpha > 1$ and $\beta < 1$, then the bound is not asymptotically sharp (and hence not asymptotically exact and not exact), but it is attainable. In this case it is valid in the domain $\mathcal{D} = \mathcal{R}_+ \times [0, a_0]$, where $a_0 := (\alpha - 1)|a|/(\alpha - \beta)$. The manifold $\mathcal{M}$ (see Definition 4) here is $\mathbf{R}_+ \times \{a_0\}$.

5. If $\alpha > 1$ and $\beta \geq 1$, then the bound has none of the properties of sharpness and exactness but is nevertheless rigorous. Its measure of asymptotic exactness is α.

In Fig. 1 we compare the exact quantity ω with the bound from case 4 with $|a| = 1$, $x = 1$ and $\alpha = 2$, $\beta = 0$ in the 3-dimensional space of the parameters $\delta_1 = \delta_c$, $\delta_2 = \delta_a$ and f. After the intersection of the surface $\omega = (\delta_c + \delta_a)/(1 - \delta_a)$ with the plane $f = 2(\delta_c + \delta_a)$ the expression for f is not a rigorous bound.

The next example shows that a bound may be asymptotically sharp without being asymtotically exact.

Example 2. Consider the equation from Example 1 together with the bound $f(\delta) := \sqrt{1 + x^2}\sqrt{\delta_c^2 + \delta_a^2}/(|a| - \delta_a)$. This bound is defined in the set $\mathcal{D} = \Omega$ but it is not asymptotically exact. At the same time it is asymptotically sharp and attainable. Indeed, we have $f(\delta) = \omega(\delta)$ at the one-dimensional manifold $\mathcal{M}$, defined via $\delta_a = |x|\delta_c < |a|$.

In Fig. 2 we show the exact quantity ω and the bound f for $|a| = x = 1$.

We have stated that a bound should at least be asymptotically sharp. This property should take place not only for a particular equation but for a large class of equations. A more desirable property of a bound is to be asymptotically exact. Here again asymptotical exactness should be established for a wide class of equations rather than for a single equation and the equations close to it. Attainability should also be proved for possibly

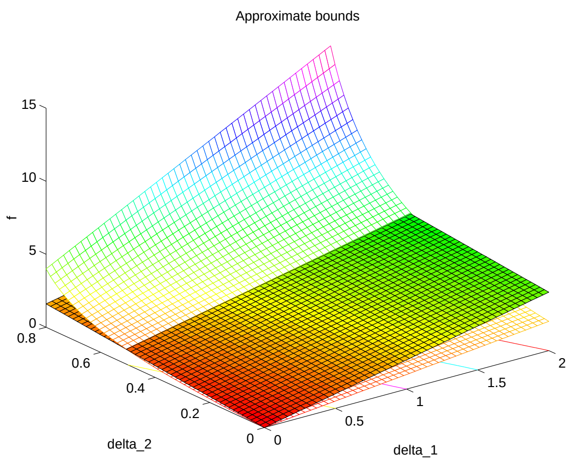

Fig. 1. An attainable bound which is not asymptotically sharp

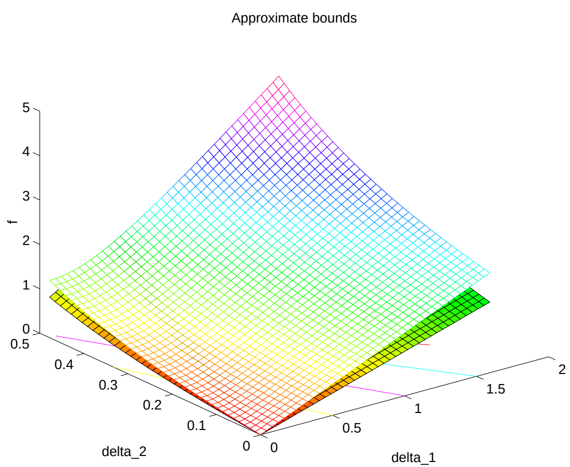

Fig. 2. An attainable asymptotically sharp bound which is not asymptotically exact

wide classes of equations, although this is a property of restricted value. Finally, the top property of a bound to be exact usually is valid, or may be proven as such, only for small classes of equations.

The concepts of asymptotical sharpness, asymptotical exactness, exactness and attainability are applicable to general matrix equations (as well as to operator equations in abstract spaces) and in particular to polynomial and fractional-polynomial equations, arising in linear algebra, control theory and other scientific applications.

3. Conservativeness of 'worst case' bounds. Consider a rigorous perturbation bound $\delta_X \leq f(\delta)$, $\delta \in \mathcal{D}$, for the problem $X = \Phi(A)$, where

$$\delta X = \Psi(\delta A, A) := \Phi(A + \delta A) - \Phi(A) \ \text{ and } \ \|\delta A\|_g \preceq \delta.$$

Since the bound is rigorous, it is also a 'worst case' perturbation bound in the following sense. The bound is valid for all perturbations $\delta A \preceq \delta$, including those for which the norm-wise perturbation δ_X in the solution is maximal. At the same time, for other perturbations, the actual perturbation δ_X may be much less (or even zero) than the bound $f(\delta)$ predicts. Thus all rigorous perturbation bounds are conservative for certain classes of particular perturbations. This is true even for exact bounds $f(\delta) = \omega(\delta)$, where $\omega(\delta) := \max\{\|\Phi(A+\delta A) - \Phi(A)\| : \|\delta A\|_g \preceq \delta\}$ is the maximal perturbation in δ_X when δA varies over the set of admissible perturbations Ω.

Next we discuss the interesting phenomenon when, for a given class $\mathcal{Q}$ of perturbations δA, the perturbation δX in the solution X is zero. For an explicit problem $X = \Phi(A)$ we have $\mathcal{Q} := \{E : \Psi(E, A) = 0\}$.

For an implicit problem, defined via an equation $F(A, X) = 0$, the set $\mathcal{Q}$ is $\mathcal{Q} := \{E \in \mathcal{D} : F(A + E, X) = 0\}$. In the generic case when the problem is regular and the partial Fréchet derivative $F_A(A, X)$ at (A, X) is surjective, the set $\mathcal{Q}$ is a manifold of dimension $\dim(\mathcal{A}) - \dim(\mathcal{X})$.

Let the matrix collection A be represented as $A = (B, C)$, where B and C are in turn matrix collections. Suppose that we may rewrite the equation $F(A, X) = 0$ in the equivalent form $G(B, X) = H(C)$, where G and H are continuous functions. If B and C are perturbed to $B + \delta B$ and $C + \delta C$, then we obtain the perturbed equation $G(B + \delta B, X + \delta X) = H(C + \delta C)$. Suppose that we have the perturbation bound $\delta_X \leq f(\beta, \gamma)$, $(\beta, \gamma) \in \Omega$ provided $\|\delta B\|_g \preceq \beta$, $\|\delta C\|_g \preceq \gamma$

If the perturbations δB, δC satisfy the additional relation $G(B+\delta B, X) = H(C+\delta C)$, then the perturbed equation has a solution $\delta X = 0$ and accordingly $\delta_X = 0$. Hence, nevertheless how good the bound $f(\beta, \gamma)$ is, it may be very conservative in this particular case.

Note that relation $G(B+\delta B, X) = H(C+\delta C)$ will be fulfilled if e.g. H is the identity operator and $\delta C = G(B + \delta B, X) - C$.

A simple example here is the linear equation $BX = C$, where B and C, X are $m \times m$ and $m \times n$ matrices, respectively, with B being non-singular, and $C \neq 0$. Supposing that $\|C^{-1}\|\|\delta B\| < 1$ and $\delta C = \delta B X = \delta B B^{-1} C$, we see that the perturbed equation $(B + \delta B)(X + \delta X) = C + \delta C$ has a unique solution $\delta X = 0$ and hence $\varepsilon_X = \|\delta X\|/\|X\| = 0$. At the same time, setting $\varepsilon_B = \|\delta B\|/\|B\|$, we have the standard perturbation bound $\varepsilon_X \leq f(\varepsilon_B) := (2\text{cond}(B)\varepsilon_B)/(1 - \text{cond}(B)\varepsilon_B)$. For ε_B approaching $1/\text{cond}(B)$ the bound $f(\varepsilon_B)$ becomes arbitrarily large while the exact estimated quantity is zero.

This effect of extreme conservatism is not generic and is destroyed in any neighbourhood of the perturbation $(\delta C, \delta B)$. Indeed, the relation $\delta C = \delta B X = \delta B B^{-1} C$ defines a m^2-dimensional subspace $\mathcal{Q}$ in the $m(n + m)$-dimensional linear space of pairs (C, B). Let $(\delta C, \delta B) \in \mathcal{Q}$ be such that $B + \delta B$ is close to a singular matrix. Then there exists a perturbation $\overline{\delta C}$ such that $(\overline{\delta C}, \delta B) \notin \mathcal{Q}$, the quantity $\|\overline{\delta C} - \delta C\|$ is small, and the relative perturbation in the solution, corresponding to the perturbation $(\overline{\delta C}, \delta B)$, is close to the bound $f(\varepsilon_B)$.

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ВЪРХУ СВОЙСТВАТА НА ПЕРТУРБАЦИОННИТЕ ГРАНИЦИ

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Въведени и изучени са някои свойства на пертурбационните граници като асимптотична острота, асимптотична точност и точност. Тези свойства са важни за пертурбационния анализ на алгебричните матрични уравнения, възникващи в линейната алгебра и теория на управлението.