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DIFFUSION APPROXIMATION OF A BRANCHING PROCESS WITH INCREASING TO INFINITY OFFSPRING VARIANCE*

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Let $Z_n^{(k)}, n = 0, 1, 2, \ldots$, be a Galton-Watson branching process with geometric offspring and large number $k \to \infty$ of ancestors. Assume that offspring mean approaches one and offspring variance increases to infinity as $k \to \infty$. We find a diffusion approximation of $Z_n^{(k)}$ under an appropriate time and space scaling. The results are extended to a process allowing immigration.

1. Introduction and Results. Diffusion models are useful as continuous time approximation to discrete processes including sums of random variables. It is well known that the Brownian motion process approximates scaled simple symmetric random walk. The theory of branching processes provides another example, namely the convergence of time and space scaled discrete time branching processes to Jiřina processes or to diffusion processes. For background and additional references on this topic see [2], [3], [5] as well as the recent paper [1].

Consider a Galton-Watson branching process $Z_n^{(k)}, n = 0, 1, 2, \ldots$, with k ancestors. Suppose that both offspring mean m_k and offspring variance σ_k^2 depend on k such that

$$m_k = 1 + \frac{\alpha_k}{k}, \qquad \text{and} \qquad \sigma_k^2 \to \sigma^2 < \infty \tag{1}$$

where $\alpha_k \to \alpha \geq 0$. Under these (and one more moment regularity condition) the following classical Feller-Jiřina limit theorem holds (see e.g. [1]). Let $k \to \infty$, then for any $t > 0$

$$\frac{Z_{[kt]}^{(k)}}{k} \xrightarrow{d} X_t,$$

where $\{X_t, t \geq 0\}$ is a diffusion process with drift αx and variance $\sigma^2 x/2$.

From now on we assume geometric offspring distribution depending on k and having pgf

$$f^{(k)}(s) = 1 - \frac{m_k(1-s)}{1 + \frac{c_k}{2m_k}(1-s)}, \tag{2}$$

where m_k is the offspring mean and $c_k = d^2 f^{(k)}(1)/ds^2$, i.e. the second factorial moment of the offspring distribution.

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Instead of (1), let for $\alpha > 0$

$$m_k = 1 + \frac{\alpha c_k}{k} \qquad \text{and} \qquad c_k = o(k) \to \infty. \tag{3}$$

Notice that, assumptions (3) imply that the offspring mean m_k approaches one as $k \to \infty$ and at the same time the offspring variance (equals $c_k(1+o(1))$) increases to infinity with k.

Theorem 1. *Let the offspring distribution be geometric given by* (2) *and* (3). *If* $k \to \infty$, *then for* $t > 0$

$$\frac{Z^{(k)}_{[tk/c_k]}}{k} \xrightarrow{d} X_t, \tag{4}$$

where $\{X_t, t \geq 0\}$ *is a diffusion process with drift* αx *and variance* $x/2$.

We shall extend the above result to a process with immigration. Consider

$$Y_n^{(k)} = Z_n^{(k)} + I_n, \qquad n = 1, 2, \ldots,$$

and $Y_0^{(k)} = k \to \infty$, where I_n is an immigration component independent from the process' reproduction. Assume that $\{I_n, n = 1, 2, \ldots\}$ are i.i.d with pgf $g(s)$ that satisfies for $s \to 1$

$$g(s) = 1 - a(1-s) + o(1-s), \tag{5}$$

where $0 < a < \infty$. The following result holds.

Theorem 2. *Let the offspring distribution be geometric given by* (2) *and* (3). *If the immigration pgf satisfies* (5) *and* $k \to \infty$, *then for* $t > 0$

$$\frac{Y^{(k)}_{[tk/c_k]}}{k} \xrightarrow{d} X_t, \tag{6}$$

where $\{X_t, t \geq 0\}$ *is a diffusion process with drift* αx *and variance* $x/2$.

2. Proofs.

Proof of Theorem 1. Taking into account (2), it is not difficult to see that the pgf of the process $Z_n^{(k)}$ is

$$Es^{Z_n^{(k)}} = \left(1 - \frac{m_k^n(1-s)}{1 + \frac{c_k}{2m_k}\frac{1-m_k^n}{1-m_k}(1-s)}\right)^k.$$

Hence,

$$E\left(e^{-(u/k)Z^{(k)}_{[tk/c_k]}}\right) = \left(1 - \frac{m_k^{[tk/c_k]}(1-e^{-u/k})}{1 + \frac{c_k}{2m_k}\frac{1-m_k^{[tk/c_k]}}{1-m_k}(1-e^{-u/k})}\right)^k.$$

On the other hand, from (3) we have as $k \to \infty$

$$m_k^{[tk/c_k]} = \left(1 + \frac{\alpha c_k}{k}\right)^{[tk/c_k]} \to e^{\alpha t}$$

and therefore, using that $1 - e^{-u/k} = u/k(1+o(1))$, as $k \to \infty$

$$\frac{m_k^{[tk/c_k]}(1-e^{-u/k})}{1 + \frac{c_k}{2m_k}\frac{1-m_k^{[tk/c_k]}}{1-m_k}(1-e^{-u/k})} \sim \frac{ue^{\alpha t}}{k(1 - \frac{u}{2\alpha}(1-e^{\alpha t}))}.$$

Now, it is not difficult to see that as $k \to \infty$

$$E\left(e^{-(u/k)Z^{(k)}_{[tk/c_k]}}\right) \sim \left(1 - \frac{ue^{\alpha t}}{k(1 - \frac{u}{2\alpha}(1 - e^{\alpha t}))}\right)^k \to \exp\left(-\frac{ue^{\alpha t}}{1 + \frac{u}{2\alpha}(e^{\alpha t} - 1)}\right),$$

the last term in the right-hand side being the Laplace transform of the one dimensional distribution of $\{X_t, t \geq 0\}$, see e.g. [2], p.68. The theorem is proved.

Proof of Theorem 2. Let us denote by $f_n^{(k)}(s)$ the pgf of $Z_n^{(k)}$. Then the pgf of the process $Y_n^{(k)}$ can be written as

$$Es^{Y_n^{(k)}} = (f_n^{(k)}(s))^k \prod_{j=0}^{n-1} g(f_j^{(k)}(s)).$$

The statement of the theorem is equivalent to (as $k \to \infty$)

$$E\left(e^{-(u/k)Y^{(k)}_{[tk/c_k]}}\right) \to \exp\left(-\frac{ue^{\alpha t}}{1 + \frac{u}{2\alpha}(e^{\alpha t} - 1)}\right).$$

On the other hand, the last equation in the proof of Theorem 1 implies for the Laplace transform of the properly time and space scaled process

$$f^{(k)}_{[\frac{tk}{ck}]}(e^{-u/k}) \to \exp\left(-\frac{ue^{\alpha t}}{1 + \frac{u}{2\alpha}(e^{\alpha t} - 1)}\right).$$

Thus, to complete the proof it is sufficient to see that as $k \to \infty$

$$\prod_{j=0}^{[tk/c_k]-1} g(f_j^{(k)}(e^{-u/k})) \to 1.$$

Applying (5) we can write

$$\prod_{j=0}^{[tk/c_k]-1} g(f_j^{(k)}(e^{-u/k})) = \exp\left(\sum_{j=0}^{[tk/c_k]-1} \log\Big(1 - a(1 - f_j^{(k)}(e^{-u/k}))(1 + o(1))\Big)\right).$$

This reduces the proof to show

$$\sum_{j=0}^{[tk/c_k]-1} \log\Big(1 - a(1 - f_j^{(k)}(e^{-u/k}))(1 + o(1))\Big)$$

$$\sim -a \sum_{j=0}^{[tk/c_k]-1} (1 - f_j^{(k)}(e^{-u/k})) \to 0$$

as $k \to \infty$ for any fixed numbers t and u. The following estimates hold

$$\begin{aligned} 0 &\leq \sum_{j=0}^{[tk/c_k]-1} (1 - f_j^{(k)}(e^{-u/k})) \\ &= \sum_{j=0}^{[tk/c_k]-1} \frac{m_k^j(1 - e^{-u/k})}{1 + \frac{c_k}{2m_k}\frac{1-m_k^j}{1-m_k}(1 - e^{-u/k})} \end{aligned}$$

$$= \sum_{j=0}^{[tk/c_k]-1} \frac{(1+\frac{\alpha c_k}{k})^j(1-e^{-u/k})}{1-\frac{k}{2\alpha m_k}\left(1-(1+\frac{\alpha c_k}{k})^j\right)(1-e^{-u/k})}$$

$$\leq \sum_{0\leq j\leq tk/c_k} \frac{\left((1+\frac{\alpha c_k}{k})^{k/c_k}\right)^{jc_k/k}(1-e^{-u/k})}{1-\frac{k}{2\alpha m_k}\left(1-\left((1+\frac{\alpha c_k}{k})^{k/c_k}\right)^{jc_k/k}\right)(1-e^{-u/k})}$$

$$\sim \frac{u}{c_k}\left\{\frac{c_k}{k}\sum_{0\leq j\leq tk/c_k}\frac{e^{\alpha jc_k/k}}{1-\frac{u}{2\alpha}(1-e^{\alpha jc_k/k})}\right\},$$

since $1-e^{-u/k}\sim u/k$, $(1+\frac{\alpha c_k}{k})^{k/c_k}\to e^{\alpha}$ and $m_k\to 1$ as $k\to\infty$.

Note that, the sum in the curly brackets is a partial sum of

$$\int_0^t \frac{e^{\alpha x}}{1-\frac{u}{2\alpha}(1-e^{\alpha x})}dx,$$

that tends to the integral as $k\to\infty$ and the integral itself is finite for any fixed t. On the other hand, the factor $u/c_k\to 0$. Therefore,

$$\lim_{k\to\infty}\sum_{j=0}^{[tk/c_k]-1}(1-f_j^{(k)}(e^{-u/k}))=0,$$

which completes the proof.

3. Final Remarks. In this note we consider the special case of geometric offspring. It is natural to seek similar results in the general offspring distribution case under the assumption for increasing to infinity variance.

A further extension would be to find diffusion approximation of branching processes with migration. In a recent article, Dyakonova [1] has obtained diffusion approximation of a branching process when the migration component is both stopped at zero and has mean zero. It seems that under the assumption for increasing to infinity variance it is possible to extend these results to processes with migration (stopped or non-stopped at zero) having positive mean.

Finally, this note can be considered as a further step in studying branching processes with increasing to infinity offspring variance, which was initiated in [3].

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LARGE СХОДИМОСТ НА РАЗКЛОНЯВАЩИ СЕ ПРОЦЕСИ С РАСТЯЩА ДИСПЕРСИЯ КЪМ ДИФУЗИОННИ

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Разглежда се прост разклоняващ се процес $Z_n^{(k)}, n = 0, 1, 2, \ldots$, започващ с голям брой частици k. Предполага се, че математическото очакване на броя на преките потомци на една частица $m_k \to 1$, а дисперсията $\sigma_k \to \infty$, когато $k \to \infty$. При тези условия е намерено скалиране на времето и пространството от състоянията на $Z_n^{(k)}$, при което процесът клони към дифузионен процес, когато $k \to \infty$. Аналогичен резултат е доказан за разклоняващи се процеси с имиграция.