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AN ACCELERATION OF ITERATIVE PROCESSES*

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A new approach for acceleration of the iterative processes for solving of nonlinear equations is proposed. There are given some results from numerical experiments, which corroborate the theoretical conclusions.

1. Introduction. Very often, solving practical problems, the solution is represented by the limit of an infinite sequence, which convergence is very slow, so slow, that sometimes we must give up the method we have chosen. In such cases there are two options. First, we can give up method we have chosen and try to find another one, for which there is a quicker convergence. The other one is to accelerate the first one. Namely on such kind of problem is devoted this short communication. The approach proposed in it is a modest contribution in this direction. There are many other surveys in connection with this subject made by: Aitken, Hartree, Henrici, Householder, Korganoff, Ostrowski, Rutishanser, Samuelson, Shanks, Steffensen, Traub and others.

2. Description and grounds of the approach. In this section we shall prove the following

Theorem 1. *Let $\varphi(x), f(x) \in C^k[a,b], \varphi(x).f'(x) \neq 0$ for every $x \in [a,b]$ and α be a root of the equation $f(x) = 0$, located in the interval (a,b). If the iterative function*

$$y = x - \varphi(x)f(x), \tag{1}$$

for computing of α has a convergence order k, i.e.

$$y'(\alpha) = y''(\alpha) = \ldots = y^{(k-1)}(\alpha) = 0, y^{(k)}(\alpha) \neq 0, \tag{2}$$

and the function

$$\psi(x) = \begin{cases} (1 - y'(\alpha))^{-1}(1 + O(\varepsilon)) & for \quad k = 1 \\ 1 + \frac{\varepsilon^{k-1} y^{(k)(\alpha)}}{k!} + O(\varepsilon^k) & for \quad k > 1 \end{cases} \tag{3}$$

where $\varepsilon = x - \alpha,\ x \in [a,b]$ and x is sufficiently close to α. Then:

a) Iterative function

$$z = x - \psi(x)\varphi(x)f(x) \tag{4}$$

has convergence order at least $k + 1$.

b) The function

$$\theta(x) = \frac{f(x)}{f(x) - f(y)} \tag{5}$$

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can be represented in the form (3).

Proof. a) Let $\delta = z - \alpha,\ v = v(x) = \varphi(x)f(x)$.

First case $k = 1$. From the assumptions it is easy to receive, that $1 - y'(\alpha) \neq 0$. Then from (4),(1) and (3) we get

$$\delta = \varepsilon - (1 - y'(\alpha))^{-1}(1 + O(\varepsilon))(v(\alpha) + \varepsilon v'(\alpha) + O(\varepsilon^2)) = O(\varepsilon^2),$$

i.e. $\delta = O(\varepsilon^2)$. Hence (4) has convergence order 2.

Second case $k > 1$. For such k, again using Taylor's formula, we obtain

$$\delta = \varepsilon - \psi v = \varepsilon - \left(1 + \frac{\varepsilon^{k-1}}{k!}y^{(k)}(\alpha) + O(\varepsilon^k)\right)\left(v(\alpha) + \varepsilon v'(\alpha) + \frac{\varepsilon^k}{k!}v^{(k)}(\alpha) + O(\varepsilon^{k+1})\right)$$

$$= \varepsilon - \left(1 + \frac{\varepsilon^{k-1}}{k!}y^{(k)}(\alpha) + O(\varepsilon^k)\right)\left(\varepsilon - \frac{\varepsilon^k}{k!}y^{(k)}(\alpha) + O(\varepsilon^{k+1})\right) = O(\varepsilon^{k+1}),$$

i.e. $\delta = O(\varepsilon^{k+1})$.

So the assertion of subsection **a)** is proved.

b) *First case* $k = 1$. The function $\theta(x)$ defined for every $x \in [a, b]$ and $x \neq \alpha$. But it is easy to verify, that $\theta(x) \to (1 - y'(\alpha))^{-1}$ when $x \to \alpha$. That's why we may put $\theta(\alpha) = (1 - y'(\alpha))^{-1}$. Further for $x \neq \alpha,\ x \in [a, b]$ and x sufficiently closed to α we have

$$\theta(x) = \frac{f(x)}{f(x) - f(y)} = \frac{f(\alpha) + \varepsilon f'(\alpha) + O(\varepsilon^2)}{f(\alpha) + \varepsilon f'(\alpha) + O(\varepsilon^2) - f(\alpha) - \varepsilon f'(\alpha)y'(\alpha) + O(\varepsilon^2)}$$

$$= (1 - y'(\alpha))^{-1}(1 + O(\varepsilon)).$$

Second case $k > 1$. In this case $\theta(\alpha) = 1$. For x sufficiently close to α, $x \neq \alpha$, we get:

$$f(x) = f(\alpha) + \varepsilon f'(\alpha) + O(\varepsilon^2) = \varepsilon f'(\alpha) + O(\varepsilon^2),$$

$$f(y) = f(\alpha) + \sum_{s=0}^{k-1} \frac{\varepsilon^{s+1}}{(s+1)!}(f'(y)y')^{(s)}_{x=\alpha} + O(\varepsilon^{k+1})$$

$$= \frac{\varepsilon^k}{k!}f'(\alpha)y^{(k)}(\alpha) + O(\varepsilon^{k+1}).$$

Hence

$$\frac{f(y)}{f(x)} = \frac{\frac{\varepsilon^{k-1}}{k!}f'(\alpha)y^{(k)}(\alpha) + O(\varepsilon^k)}{f'(\alpha) + O(\varepsilon)}$$

$$\frac{\frac{\varepsilon^{k-1}}{k!}y^{(k)}(\alpha) + O(\varepsilon^k)}{1 + O(\varepsilon)} = \frac{\varepsilon^{k-1}}{k!}y^{(k)}(\alpha) + O(\varepsilon^k).$$

Then

$$\theta(x) = \frac{f(x)}{f(x) - f(y)} = \frac{1}{1 - \frac{f(y)}{f(x)}} = \frac{1}{1 - \left(\frac{\varepsilon^{k-1}}{k!}y^{(k)}(\alpha) + O(\varepsilon^k)\right)}$$

$$= 1 + \frac{\varepsilon^{k-1}}{k!}y^{(k)}(\alpha) + O(\varepsilon^k).$$

The assertion of subsection **b)** is proved.

3. Examples. a) Let $y = x - cf(x)$, where $c \neq 0$ and $c \neq \dfrac{1}{f'(\alpha)}$. This function has convergence order 1 $(y'(\alpha) \neq 1)$. The corresponding function

$$z = x - cf(x).\theta(x) = x - \frac{cf^2(x)}{f(x) - f(y)} \tag{6}$$

has convergence order 2.

b) To iterative function

$$y = x - \frac{(x-a)f(x)}{f(x) - f(a)} \quad (Regula\ falsi\ method)$$

of order 1 it corresponds thr iterative function

$$z = x - \frac{(x-a)f(x)}{f(x) - f(a)}.\theta(x) = x - \frac{(x-a)f^2(x)}{(f(x) - f(a))(f(x) - f(y))} \tag{7}$$

of order 2.

c) To iterative function

$$y = x - \frac{f(x)}{f'(x)} \quad (\text{Newton's method})$$

of order 2 it corresponds the iterative function

$$z = x - \frac{f(x)}{f'(x)}.\theta(x) = x - \frac{f^2(x)}{f'(x)(f(x) - f(y))} \quad (\text{The Newton} - \text{Secant method}) \tag{8}$$

of order 3.

4. Remarks and Comments.

• Let I_y, I_z be efficiency numbers EEF (see [1], chapter 1) of iterative functions y, z. Then it is easy to check, that $I_y \leq 1$ implies $I_y \leq I_z$.

• The approach, proposed here, can be applied many times, if the corresponding conditions for the iterative functions and ψ are fulfilled.

• The iterative function $z = x - \varphi(x)f(x)\theta(x)$ is a special case of a family of iterative processes discussed in [1] (pp.145–152), but there the discussion is not connected with acceleration.

• There are approaches with larger coefficients of acceleration, but they need more computational time for one iteration [2], [3], [4].

5. Numerical experiments. We have done numerical experiments for various functions and initial points. All programms were written in MATLAB. We compare four iterative procedures for computing the root of nonlinear equations.

The computations are for process:

$$x_n = \varphi(x_{n-1}),\ n = 1, 2, \ldots$$

We use the following stop stoping criteria for computer programs:

1. $|x_{n+1} - x_n| < \varepsilon$;

2. $|f(x_{n+1})| < \varepsilon$;

Example 1. Let $f(x) = \dfrac{x^3 - 1}{3}$ and $x_0 = 1.5$ is the initial approximation of the root $\alpha = 1$. For $\varepsilon_1 = 1e - 16$; $\varepsilon_1 = 1e - 18$; $\varepsilon_1 = 1e - 4$.

Table 1.

ε_i	*Number of iterations* n				*Execution time* t			
	MN	φ_1	φ_2	φ_3	MN	φ_1	φ_2	φ_3
ε_1	6	4	5	4	3.42	2.24	4.79	3.02
ε_2	5	3	4	3	2.87	1.69	3.92	2.29
ε_3	4	3	3	2	2.28	1.70	3.00	1.52

Example 2. Let $f(x) = \sqrt{(x-4)^2+2} - x^3 - 9$ and $x_0 = -1$ is the initial approximation of the root $\alpha \approx -1.492987028143630$.

Table 2.

ε_i	*Number of iterations* n				*Execution time* t			
	MN	φ_1	φ_2	φ_3	MN	φ_1	φ_2	φ_3
ε_1	6	10	6	3	4.20	6.66	6.86	3.69
ε_2	5	9	4	3	3.48	5.96	4.65	2.75
ε_3	4	8	3	3	2.78	5.31	3.55	2.73

We introduce notations: x_0 – an initial point; n – the number of iterations; t - the execution time for 10000 times execution; MN- Newton's iterative formula; φ_3-Iterative formulae (6) for $c = 1$; φ_2- Iterative formulae (7) ; φ_3- Iterative formulae (8).

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УСКОРЯВАНЕ НА ИТЕРАЦИОННИ ПРОЦЕСИ

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Предложен е нов подход за ускоряване на итерационни процеси за решаване на нелинейни уравнения. Дадени са резултатите от проведени експерименти, които потвърждават теоретичните изводи.