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CURVES AND SURFACES OF LEARNING

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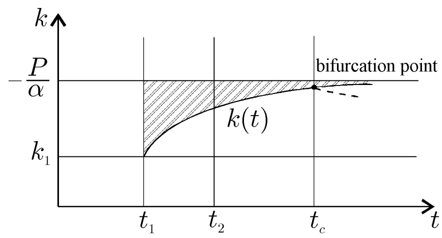
By means of the package MATHEMATICA it is shown the possibility to generate curves of learning, which present the processes of progress and fall in the preparation of students for Mathematical Olympiads. Two new notions are introduced, namely "fan" and "surface" of learning, which are with methodological significance to realize comparison and control.

Introduction. Psychology of learning is central in Pedagogical psychology. According to G. Piryov [1, p. 56] "the most visual way to express the progress of improvement is to use curves of learning". In fact the curves of learning considered by Piryov and other authors are curves of receptivity but we stick to the first notion due to its general recognition.



What is shown in Fig. 1 (see also [2]) is the change of the training level of a student in a course of preparation for a Mathematical Olympiad. The main peculiarities of the stage under consideration are connected with the "negative acceleration" of Thorndick [3], the ascending direction of each learning, the appearance of typical points, and s.o. Plateau and bifurcation are of practical importance. The point t_c of critical time is important too. It marks off a fall in a saturation zone. Its establishment in time is a prerequisite to avoid eventual bifurcations and their undesirable consequences. Laboratory derivation of the curves of learning gives a possibility to prognosticate the moment t_c. The working hypothesis is that t_c appears when the area of the domain which boundaries are the restriction of the curve of learning on the interval $[t_1; t_c]$, the lines $t = t_1$, $t = t_c$ and $k = -\dfrac{P}{\alpha}$, is about 90% of the corresponding area for the interval $[t_1; \infty)$. The applicability of the approach follows from the theoretical result in the sequel.

Main theoretical result. Let us consider the following linear differential equation with constant coefficients:

$$\frac{dk}{dt} = \alpha k + P \tag{1}$$

with initial condition $k(t_1) = k_1$. When $k_1 \neq -\dfrac{P}{\alpha}$ the solution of the equation (see Fig. 1) is $k(t) = \dfrac{(\alpha k_1 + P)\, e^{\alpha(t-t_1)} - P}{\alpha}$. In the case $k_1 = -\dfrac{P}{\alpha}$ we get $k\,(t) = -\dfrac{P}{\alpha}$. If $k_1 \neq -\dfrac{P}{\alpha}$, then the line $k = -\dfrac{P}{\alpha}$ is a horizontal asymptote to the solution. Fig. 2 shows the graphical representation of (1) when $\alpha = \ln(0,2)$, $P = 20$ and $k_0 = k(0) = 1$. The dotted line stands for the horizontal asymptote $k = -\dfrac{20}{\ln(0,2)}$.

Fig. 2

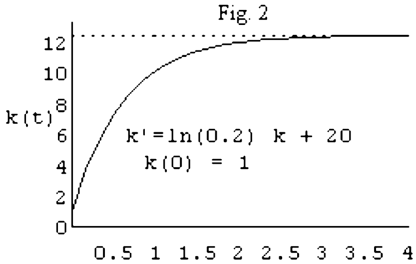
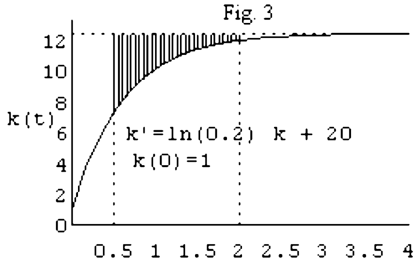


Fig. 3



We will find the area $S(t_1, t_2)$ in the general case when the boundary of the domain is formed by the restriction of the solution of (1) and the asymptote on the interval $[t_1, t_2]$. Since

$$S(t_1, t_2) = \int_{t_1}^{t_2} -\frac{P}{\alpha} - \frac{(\alpha k_0 + P)\, e^{\alpha(t-t_0)} - P}{\alpha} dt,$$

then

$$S(t_1, t_2) = \frac{e^{-\alpha t_0}\,(e^{\alpha t_1} - e^{\alpha t_2})\,(P + \alpha k_0)}{\alpha^2},$$

where $k(t_0) = k_0$. If $t_2 \to +\infty$ in the last equality, then

$$S(t_1, \infty) = \frac{e^{\alpha}\,(t_1 - t_0)\,(P + \alpha k_0)}{\alpha^2}. \tag{2}$$

Further, let $0 < m < 1$ be a fixed number. We are interested in finding t_2 such that $S(t_1, t_2) = mS(t_1, \infty)$. The solution of the last equation with respect to m gives

$$t_2 = \frac{1}{\alpha} ln\,(1 - m) + t_1. \tag{3}$$

We deduce that t_2 depends on the coefficient α in front of k only and does depend neither on the initial condition nor on the coefficient P. If $m = 0,9$ we get the moment when 90% of the area of the domain under consideration is reached. Thus, the following assertion could be formulated:

Theorem. *The appearance of a critical moment t_c for the curve of learning* (1) *does not depend on the initial condition. It depends on the student individual characteristics only.*

The significance of this theoretical result is important. The starting points of the curves of learning are established by measuring the student's level of potentialities at the beginning of the preparatory stage. All measures are connected with problem solving and tests. Thus, they are accidental since they depend on various factors, the quality of the problems included. On the other hand, the appearance of plateau and saturation zone is influenced by learning and not by initial conditions. In such a way the result registers an agreement between the real situation and the theoretical expectation.

Practical application. In Fig. 3 the hatched part represents 90% of the area of the domain which boundary is formed by the restriction of the asymptote and the curve of learning from Fig. 1 on the interval $[0,5;\infty)$. In this case $t_1 = 0,5$ and $t_2 = 2$. The results are obtained using the package MATHEMATICA. The next Fig. 4 is connected with the curves of learning of another student whose individual characteristic is $\alpha = \ln(0,5)$, while the preparation P is kept the same. Two curves are considered. The thick one starts from the point $k(0) = 20$, while the starting point of the dotted one is $k(0) = 0$. This corresponds to the extreme cases when the student has received the maximum of 20 points and the minimum of 0 points at the entering test, respectively. The final result is one and the same, i.e. in both cases the critical point is $t_c = 4,4$. Analogous conclusions follow from Fig. 5, where the curves of learning of the same student are shown ($\alpha = \ln(0,5)$). Now, more starting points are considered. The asymptote $k = -\dfrac{20}{\ln(0,5)}$ is presented by a dotted line. If the starting points are above the asymptote, then the corresponding curves are with opposite slope. This means that in case of a high score at the entering exam and a preparation, which is below the level of the student, the participation in the concrete preparation makes no sense at all. In fact, such starting points are unattainable practically and their significance is only theoretical. The representation in Fig. 5 looks like a fan and for this reason we will note it in this way. Thus, what is shown in Fig. 5 is **the fan** of Lozan Ivanov who is a student in the National Mathematical School.

The curve of learning of a hypothetical student is shown in Fig. 6. The different parts of the curve correspond to different stages of preparation in which the student participates with 5 individual characteristics, corresponding to 5 different school years.

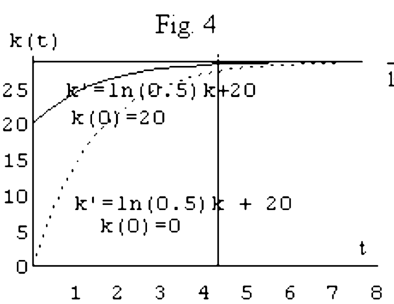


Fig. 4

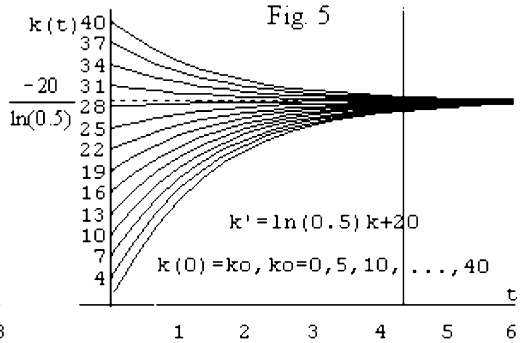


Fig. 5

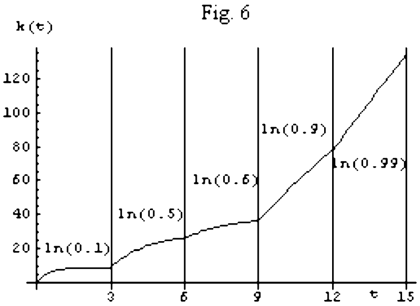


Fig. 6

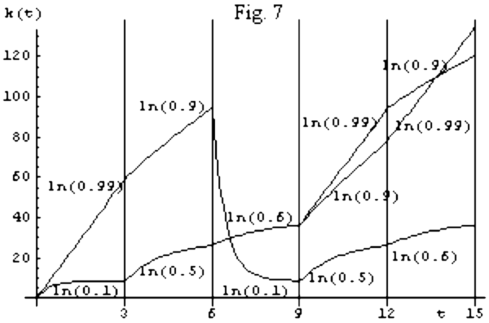


Fig. 7

Each of the characteristics is essential for a group of students G_i , i = 1, 2, 3, 4, 5. A special methodology to create these groups is proposed by the first author of the present paper. Some of the details are as follows. During the 2001/2002 school year 10 mathematical competitions that take place in Bulgaria have been observed. Using a special scale the scores of the participating students have been equalized to the score, which was sufficient for a gold medal during the last International Olympiad. Those in group G_1 have obtained 10 out of 10 minimally sufficient scores for a gold medal. Thus, the probability for them to win a gold medal at the next Olympiad is 1 and the control parameter of the group is defined by the number $\ln(0,99)$ (in stead of 1 it is taken 0,99, where the function $\ln x$ is well defined). In the same way those who have obtained 9 out of 10 minimally sufficient scores for a gold medal have been included in group G_2. Now, the probability for a gold medal is 0,9 and the control parameter of the group is $\ln(0,9)$. Similarly, the control parameters of G_3, G_4 and G_5 are $\ln(0,6)$, $\ln(0,5)$ and $\ln(0,1)$, respectively.

The curve in Fig. 6 shows the ideal case when the hypothetical student passes from one group to another smoothly. The graphical representation corresponds to a tendency of improvement. On the contrary, chaotic passing is possible as well and this could be seen in Fig. 7, where a comparison of different curves is presented. It is obvious from the figure that at the end of the 5-years period the highest result is achieved under smooth passing, which means a gradual increase of the individual characteristics. The theoretical conclusion is in good agreement with the practice. An interesting example is connected with the student Ilya Tsekov from the Sofia Mathematical School. In 2000 he won a bronze medal at the Balkan Olympiad in Moldova. According to his presentation during the 2000/2001 school year he passed from group G_2 to group G_3 unexpectedly. Later, during the 2001/2002 school year Ilya Tsekov came back to group G_2. He won a silver medal at the Balkan Olympiad in Turkey and again a bronze medal at the International Olympiad in Scotland. The achievements of this student are very good indeed but unfortunately they are not the highest, i.e. he did not win a gold medal at the International Olympiad.

Fig. 8 and Fig. 9 are dedicated to surfaces of learning. We are not familiar if other authors have used such surfaces. According to us their application guarantees a better visualness and more convenient possibilities for comparison. For example, the surface of learning from Fig. 8 belongs to a hypothetical student whose individual characteristic is equal to $\alpha = \ln(0,2)$. If the preparation $P = 25$ is fixed, the surface presents a continuous

Fig. 8

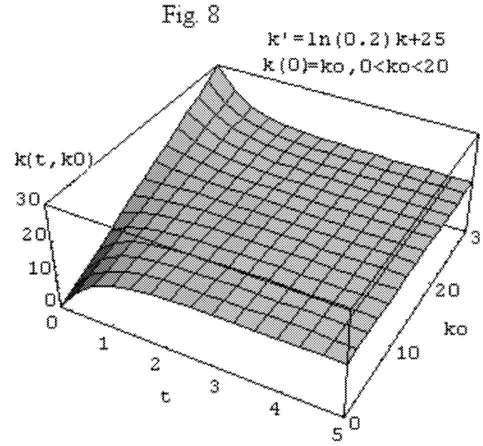
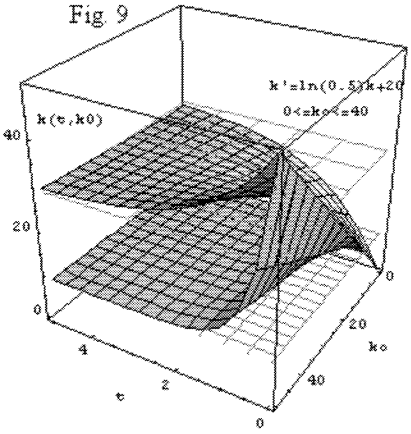


Fig. 9



change of the curves of learning when k_0 varies in the interval $[0;30]$. In 3D representation the asymptote is replaced by an asymptotic plane, which in the case under consideration is parallel to the plane tOk_0. Fig. 9 shows the surfaces of learning of two students. The preparation $P = 20$ is fixed again, while the starting points change, i.e. the parameter k_0 varies continuously in the interval $[0;40]$. Both asymptotic planes are well expressed.

Conclusion. Various investigations of educational processes are possible by the use of curves, fans and surfaces of learning. Particularly, it is well known the pedagogical significance of the curves. As stated by G. Piryov [1, p. 70 - 165] "the curves of learning are not only useful instruments to express the results of each learning but also give means to control these results". On the other hand teachers could use curves, fans and surfaces of learning to compare the achievements of their students and to check their own approaches and methods in teaching.

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КРИВИ И ПОВЪРХНИНИ НА УЧЕНЕ

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С помощта на пакета MATHEMATICA е показана възможността за генериране на криви на учене, които представят процесите на възход и спад при подготовка на ученици за математически олимпиади. Въведени са две нови понятия – „ветрило“ и „повърхнина“ на учене, които имат методическо значение за осъществяване на сравнения и контрол.