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# PERTURBATION BOUNDS FOR GENERAL COUPLED MATRIX RICCATI EQUATIONS*

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In this paper we derive perturbation bounds for general real algebraic continuous-time coupled matrix Riccati equations related to modern control theory.

**Introduction.** In this paper we present a complete perturbation analysis of real algebraic continuous-time coupled matrix Riccati equations (CAMRE). Equations of this type arise in modern control theory. The results obtained below are based on the technique proposed in [2].

Throughout the paper we use the following notations: $\mathbb{R}^{m\times n}$ – the space of $m\times n$ real matrices and $\mathcal{R} = \mathbb{R}^{n\times n}$; $\mathbb{R}^m = \mathbb{R}^{m\times 1}$; $\mathbb{R}_+ = [0,\infty)$; $A^\top$ – the transpose of the matrix $A$; $\preceq$ – the component-wise (partial) order relation on $\mathbb{R}^{m\times n}$; $\mathrm{vec}(A) \in \mathbb{R}^{mn}$ – the column-wise vector representation of the matrix $A \in \mathbb{R}^{m\times n}$; Mat(L)$\in \mathbb{R}^{pq\times mn}$ – the matrix representation of the linear matrix operator $\mathbf{L}$:$\mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ $\mathrm{vec}(\mathbf{L})(X))$ =Mat($\mathbf{L}$)vec($X$) for all $X \in \mathbb{R}^{m\times n}$; $I_n$ – the unit $n\times n$ matrix; $A\otimes B = [a_{pq}B]$ – the Kronecker product of the matrices $A = [a_{pq}]$ and $B$; $\|\cdot\|_2$ – the Euclidean norm in $\mathbb{R}^m$ or the spectral (or 2-) norm in $\mathbb{R}^{m\times n}$; $\|\cdot\|_\mathrm{F}$ – the Frobenius (or F-) norm in $\mathbb{R}^{m\times n}$; $\|\cdot\|$ – a replacement of either $\|\cdot\|_2$ or $\|\cdot\|_\mathrm{F}$; $\mathrm{rad}(A)$ – the spectral radius of the square matrix $A$; $\det(A)$ – the determinant of the square matrix $A$.

The space of linear operators $\mathcal{L}_1 \to \mathcal{L}_2$, where $\mathcal{L}_1$, $\mathcal{L}_2$ are linear spaces, is denoted by $\mathbf{Lin}(\mathcal{L}_1,\mathcal{L}_2)$. We also use the abbreviation $\mathbf{Lin}$=$\mathbf{Lin}(\mathcal{R},\mathcal{R})$.

**Problem statement.** Consider the system of real continuous-time CAMRE

$$
\begin{aligned}
G_1(X_1,X_2,P_1) &:= A_{11}X_1 + X_1B_{11} + C_1 + X_1D_{11}X_1 + A_{12}X_2 \\
&\quad + X_2B_{12} + X_2D_{12}X_2 + X_1E_1X_2 + X_2F_1X_1 = 0, \\
G_2(X_1,X_2,P_2) &:= A_{21}X_1 + X_1B_{21} + C_2 + X_1D_{21}X_1 + A_{22}X_2 \\
&\quad + X_2B_{22} + X_2D_{22}X_2 + X_2E_2X_1 + X_1F_2X_2 = 0,
\end{aligned}
\tag{1}
$$

where $X_i \in \mathcal{R}$ are the unknown matrices $A_{ij}$, $B_{ij}$, $C_i$, $D_{ij}$, $E_i, F_i \in \mathcal{R}$, $i,j = 1,2$ are given matrix coefficients and $P_i := (A_{ir}, B_{ir}, C_i, D_{ir}, E_i, F_i) \in \mathcal{R}^9$, $r = 1,2$.

We set $P := (P_1,P_2) = (A_{11},A_{12},B_{11},B_{12},C_1,D_{11},D_{12},E_1,A_{21},A_{22},B_{21},B_{22},C_2,$ $D_{21},D_{22},E_2,F_2)$. Denote the individual matrix members of $P$ as $\mathcal{E}_1,\dots,\mathcal{E}_{18}$.

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$P =: (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, \mathcal{E}_9, \mathcal{E}_{10}, \mathcal{E}_{11}, \mathcal{E}_{12}, \mathcal{E}_{13}, \mathcal{E}_{14}, \mathcal{E}_{15}, \mathcal{E}_{16}, \mathcal{E}_{17}, \mathcal{E}_{18}) \in \mathcal{R}^{18}$. The *generalized norm* of the matrix 18-tuple $P$ is the vector $|||P||| := [||\mathcal{E}_1||_{\mathrm{F}}, \ldots, ||\mathcal{E}_{18}||_{\mathrm{F}} \in \mathbb{R}_+^{18}$.

Denote $G := (G_1, G_2)$. Then the system (1) may be written as $G(X, P) = 0$, where $G$ is considered as a mapping $\mathcal{R}^{20} \to \mathcal{R}^2$, or $\mathbb{R}^{n\times 2n} \times \mathcal{R}^{18} \to \mathbb{R}^{n\times 2n}$.

Denote by $G_X(X, P)(\cdot) \in \mathbf{Lin}(\mathcal{R}^2, \mathcal{R}^2)$ the partial Fréchet derivative of $G$ in $X$, computed at the point $(X, P)$.

We assume that the system (1) has a solution $X = (X_1, X_2) \in \mathcal{R}^2$ such that the partial Fréchet derivative $G_X(X, P)(\cdot)$ of $G$ in $X$ at the point $(X, P)$ is invertible. Then the solution $X$ is isolated.

Let the matrices from $P_i$ be perturbed as $A_{ij} \mapsto A_{ij} + \delta A_{ij}$, etc. Denote by $P_i + \delta P_i$ the perturbed collection $P_i$, in which each matrix $Z \in P_i$ is replaced by $Z + \delta Z$ and let $\delta P = (\delta P_1, \delta P_2)$. Then the perturbed version of the equation is $G(X + \delta X, P + \delta P) = 0$. Since the operator $G_X$ is invertible, the perturbed equation has a unique isolated solution $Y = X + \delta X \in \mathbb{R}^2$ in the neighbourhood of $X$ if the perturbation $\delta P$ is sufficiently small.

Denote by $\delta := [\delta_1^\top, \delta_2^\top]^\top \in \mathbb{R}_+^{18}$, where $\delta_i := [\delta_{A_{ir}}, \delta_{B_{ir}}, \delta_{C_i}, \delta_{D_{ir}}, \delta_{E_i}, \delta_{F_i}]^\top \in \mathbb{R}_+^9$, $r = 1, 2$, the vector of absolute Frobenius norm perturbations $\delta_Z := \|\delta Z\|_F$ in the data matrices $Z \in P$.

The perturbation problem for CAMRE (1) is to find bounds $\delta_{X_i} \le f_i(\delta)$, $\delta \in \Omega \subset \mathbb{R}_+^{18}$, for the perturbations $\delta_{X_i} := \|\delta X_i\|_{\mathrm{F}}$. Here $\Omega$ is a certain set and $f_i$ are continuous functions, non-decreasing in each of their arguments and satisfying $f_i(0) = 0$. The inclusion $\delta \in \Omega$ guarantees that the perturbed CAMRE has a unique solution $Y = X + \delta X$ in a neighbourhood of the unperturbed solution $X$ such that the elements of $\delta X_1$, $\delta X_2$ are analytic functions of the elements of the matrices $\delta Z$, $Z \in P$, provided $\delta$ is in the interior of $\Omega$.

**Local perturbation analysis.** Since $G_i(X, P_i) = 0$, $i = 1, 2$, then the perturbed equations may be written as $G_i(X+\delta X, P_i+\delta P_i) = G_{i,X}(X, P_i)(\delta X)+\sum_{Z\in P_i} G_{i,Z}(\delta Z)+ H_i(\delta X, \delta P_i) = 0$, where $G_{i,X}(X, P_i)(Y) = G_{i,X_1}(X, P_i)(Y_1) + G_{i,X_2}(X, P_i)(Y_2)$ are the partial Fréchet derivative of $G_i(X, P_i)$ in $X$ at $(X, P)$ and $G_{i,Z}(\cdot) := G_{i,Z}(X, P_i)(.) \in \mathbf{Lin}$, $Z \in P_i$, are the Fréchet derivatives of $G_i(X, P_i)$ in the matrix argument $Z$, evaluated at the point $(X, P_i)$. The matrix expressions $H_i(\delta X, \delta P_i) = O\left(\|[\delta X, \delta P_i]\|^2\right)$, $\delta X \to 0$, $\delta P_i \to 0$, contain second and higher order terms in $\delta X$, $\delta P_i$. In fact, for $Y = (Y_1, Y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} H_1(Y, \delta P_1) \;=\;& X_1(\delta D_{11} Y_1 + \delta E_1 Y_2) + (Y_1 \delta D_{11} + Y_2 \delta F_1) X_1 \\ &+ X_2(\delta F_1 Y_1 + \delta D_{12} Y_2) + (Y_1 \delta E_1 + Y_2 \delta D_{12}) X_2 \\ &+ \delta A_{11} Y_1 + Y_1 \delta B_{11} + \delta A_{12} Y_2 + Y_2 \delta B_{12} \\ &+ Y_1(D_{11} + \delta D_{11}) Y_1 + Y_1(E_1 + \delta E_1) Y_2 \\ &+ Y_2(F_1 + \delta F_1) Y_1 + Y_2(D_{12} + \delta D_{12}) Y_2 \end{aligned} \tag{2}$$

and

$$\begin{aligned} H_2(Y, \delta P_2) \;=\;& X_1(\delta D_{21} Y_1 + \delta F_2 Y_2) + (Y_1 \delta D_{21} + Y_2 \delta E_2) X_1 \\ &+ X_2(\delta E_2 Y_1 + \delta D_{22} Y_2) + (Y_1 \delta F_2 + Y_2 \delta D_{22}) X_2 \\ &+ \delta A_{21} Y_1 + Y_1 \delta B_{21} + \delta A_{22} Y_2 + Y_2 \delta B_{22} \\ &+ Y_1(D_{21} + \delta D_{21}) Y_1 + Y_1(F_2 + \delta F_2) Y_2 \\ &+ Y_2(E_2 + \delta E_2) Y_1 + Y_2(D_{22} + \delta D_{22}) Y_2. \end{aligned} \tag{3}$$

The linear operator $G_X(X,P)(\cdot) = G_{i,X}(X,P_1)(.), G_{2,X}(X,P_2)(.))$ is calculated via the operators $G_{i,X_j}(X,P_i)(\cdot) = \mathbf{L}_i(\cdot),\ G_{i,X_j}(X,P_i)(\cdot) = \mathbf{L}_{ij}(\cdot),\ i,j = 1,2$. A direct calculation gives

$$G_{i,X_j}(X,P_i)(Z) = S_{ij}Z + ZT_{ij}, \tag{4}$$

where $S_{ii} = A_{ii} + X_iD_{ii} + X_jF_i$, $S_{ij} = A_{ij} + X_jD_{ij} + X_iE_i$, $T_{ii} = B_{ii} + D_{ii}X_i + E_iX_j$, $T_{ij} = B_{ij} + D_{ij}X_j + F_iX_i$.

Further on we use the following abbreviations for the partial Fréchet derivatives of $G$ and $G_i$ $\mathbf{L}(\cdot) := G_X(X,P)(\cdot) \in \mathbf{Lin}(\mathbb{R}^2,\mathbb{R}^2)$, $\mathbf{L}_i(\cdot) := G_{i,X}(X,P_i)(\cdot) \in \mathbf{Lin}(\mathbb{R}^2,\mathbb{R})$, $\mathbf{L}_{ij}(\cdot) := G_{i,X_j}(X,P_i)(\cdot) \in \mathbf{Lin}(\mathbb{R},\mathbb{R})$. Thus $G_X(X,P)(Y) = (\mathbf{L}_1(Y),\mathbf{L}_2(Y)) = (\mathbf{L}_{11}(Y_1)$ $+\mathbf{L}_{12}(Y_2),\ \mathbf{L}_{21}(Y_1) + \mathbf{L}_{22}(Y_2))$. Applying the vec operation to the pair $G_X(X,P)(Y)$ we find that the matrix representation of the linear operator $\mathbf{L}(\cdot)$ is $L$ :=Mat$(\mathbf{L}(\cdot))$ $= \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \in \mathbb{R}^{2n^2\times 2n^2}$, where $L_{ij} := I_n \otimes S_{ij} + T_{ij}^\top \otimes I_n$. Here $L_{ij} \in \mathbb{R}^{n^2\times n^2}$ is the matrix of the operator $\mathbf{L}_{ij}(\cdot)$.

We also have $G_{i,A_{ij}}(Z) = ZX_j$, $G_{i,B_{ij}}(Z) = X_jZ$, $G_{i,C_i}(Z)=Z$, $G_{i,D_{ij}}(Z)=X_jZX_j$, $G_{i,E_i}(Z) = X_iZX_j$, $G_{i,F_i}(Z) = X_jZX_i$.

The inverse $\mathbf{M}(\cdot) := \mathbf{L}^{-1}(\cdot) \in \mathbf{Lin}(\mathbb{R}^2 \times \mathbb{R}^2)$ of the operator $\mathbf{L} = G_X(X,P)(\cdot)$ may be represented as $\mathbf{L}^{-1}(\cdot) = (\mathbf{M}_1(\cdot),\mathbf{M}_2(\cdot))$, where, for $Z := (Z_1,Z_2) \in \mathbb{R}^2$, $\mathbf{M}_i(Z)$ $= \mathbf{M}_{i1}(Z_1) + \mathbf{M}_{i2}(Z_2),\ M_{ij}(\cdot) \in \mathbf{Lin}$. Hence

$$\delta X = -\mathbf{M}(W_1(\delta X,\delta P_1), W_2(\delta X,\delta P_2)), \tag{5}$$

where $W_i(Y,\delta P_i) := \sum_{Z\in P_i} G_{i,Z}(\delta Z) + H_i(Y,\delta P_i)$. In this way $\delta X_i = -\sum_{j=1}^{2} \mathbf{M}_{ij}(W_j(\delta X,\delta P_j))$, which gives

$$\delta X_i = -\sum_{j=1}^{2}\sum_{Z\in P_j} \mathbf{M}_{ij}\circ G_{j,Z}(\delta Z) - \sum_{j=1}^{2}\mathbf{M}_{ij}(H_j(\delta X,\delta P_j)),\ i = 1,2. \tag{6}$$

Therefore $\delta_{X_i} \le \sum_{j=1}^2 \sum_{Z\in P_j} K_{ij,Z}\delta_Z + O(\|\delta\|^2),\ \delta \to 0$, where $K_{ij,Z} := \|\mathbf{M}_{ij} \circ G_{j,Z}\|_{\mathbf{Lin}},\ i,j = 1,2$, is the *absolute condition number* of the solution component $X_i$ with respect to the matrix $Z \in P_j$. Here $\|.\|_{\mathbf{Lin}}$ is the induced norm in the space **Lin** of linear operators $\mathbb{R} \to \mathbb{R}$.

If $X_i \ne 0$, estimates in terms of relative perturbations are $\varepsilon_{X_i} \le \sum_{j=1}^2 \sum_{Z\in P_i} k_{ij,Z}\varepsilon_Z$ $+O(\|\delta\|^2),\ \delta \to 0$, where the quantity $k_{ij,Z} := K_{ij,Z}\|Z\|_{\mathrm{F}}$
$|X_i\|_{\mathrm{F}},\ i,j = 1,2$, is the *relative condition number* of the solution component $X_i$ with respect to the matrix coefficient $0 \ne Z \in P_j$.

The calculation of the condition numbers $K_{ij,Z}$ is straightforward when the Frobenius norm is used in $\mathbb{R}$. Indeed, let $U \in \mathbf{Lin}$. Then $\|U\|_{\mathbf{Lin}} := \max\|\|U(Z)\|_{\mathrm{F}} : \|Z\|_{\mathrm{F}} = 1\}$ $= \max\{\|\mathrm{vec}\,(U(Z))\|_2 : \|\mathrm{vec}\,(Z)\|_2 = 1\} = \max\{\|\mathrm{Mat}\,(U)\mathrm{vec}\,(Z)\|_2 : \|\mathrm{vec}\,(Z)\|_2 = 1\}$ $= \|\mathrm{Mat}\,(U)\|_2 = \sigma_{\max}(\mathrm{Mat}\,(U))$, where $\sigma_{\max}(A)$ is the maximum singular value of the matrix $A$.

Let $L_{i,Z} \in \mathbb{R}^{n^2\times n^2}$ be the matrix of the operator $G_{i,Z} \in \mathbf{Lin}$. Then a direct calculation yields $L_{i,A_{ij}} = X_j^\top \otimes I_n$, $L_{i,B_{ij}} = I_n \otimes X_j$, $L_{i,C_i} = I_{n^2}$, $L_{i,D_{ij}} = X_j^\top \otimes X_j$, $L_{i,E_i}$ $= X_j^\top \otimes X_i$, $L_{i,F_i} = X_i^\top \otimes X_j$.

Let the matrix representation of the operator $\mathbf{M}(\cdot) = G_X^{-1}(X,P)(\cdot) \in \mathbf{Lin}(\mathcal{R}^2,\mathcal{R}^2)$ be denoted as $M := \mathrm{Mat}\,(\mathbf{M}) = L^{-1} := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, $M_{ij} \in \mathbb{R}^{n^2\times n^2}$. Having in mind the expressions for the matrix representations $L_{i,Z}$ of the linear matrix operators $\mathbf{L}_{i,Z}$, the absolute condition numbers are calculated from $K_{ij,Z} = \|M_{ij}L_{j,Z}\|_2,\ Z \in P_j,\ i,j = 1,2.$

The operator equations (6) for the perturbation $\delta X_i$ may be written in a vector form as

$$\mathrm{vec}\,(\delta X_i) = \sum_{j=1}^{2} \sum_{Z\in P_j} N_{i,Z}\mathrm{vec}\,(\delta Z) - \sum_{j=1}^{2} M_{ij}\mathrm{vec}\,(H_j(\delta X, \delta P_j)),\ i = 1,2. \tag{7}$$

Note that the bounds $\mathrm{est}_i^{(1)}(\cdot)$ are linear functions in the perturbation vector $\delta \in \mathbb{R}^{18}$.

Relations (7) give a perturbation bound $\delta_{X_i} \le \mathrm{est}_i^{(2)}(\delta) + O(\|\delta\|^2),\ \delta \to 0$, where $\mathrm{est}_i^{(2)}(\delta) := \|N_i\|_2\|\delta\|_2$ and $N_i := [N_{i,1}, N_{i,2}] \in \mathbb{R}^{n^2\times 18n^2}$, $N_{i,j} := [N_{i,A_{rj}}, N_{i,B_{rj}},\ N_{i,C_j},$ $N_{i,D_{rj}}, N_{i,E_i}, N_{i,F_i}] \in \mathbb{R}^{n^2\times 9n^2}$, $i,r = 1,2$. We also have $\delta^2_{X_i} = \mathrm{vec}^\top(\delta X_i)\mathrm{vec}\,(\delta X_i)$ $= \eta^\top N_i^\top N_i\eta + O(\|\delta\|^2),\ \delta \to 0$, where $\eta := [\mathrm{vec}^\top(\delta A_{11}), \mathrm{vec}^\top(\delta A_{12}), \mathrm{vec}^\top(\delta B_{11}), \ldots,$ $\mathrm{vec}^\top(\delta F_2)]^\top \in \mathbb{R}^{18n^2}$. We shall represent the matrix $N_i^\top N_i \in \mathbb{R}_+^{18n^2\times 18n^2}$ as a $18\times 18$ block matrix with $n^2\times n^2$ blocks as follows. Let the $n^2 \times n^2$ blocks of $N_i$ be denoted as $\widehat{N}_{i,k},\ k = 1,\ldots,18$, i.e., $N_i = \left[\widehat{N}_{i,1}, \widehat{N}_{i,2}, \ldots, \widehat{N}_{i,18}\right]$, $\widehat{N}_{i,k} \in \mathbb{R}^{n^2\times n^2}$, where $\widehat{N}_{i,1}$ $:= N_{i,A_{11}}$, $\widehat{N}_{i,2} := N_{i,A_{12}}$, $\widehat{N}_{i,3} := N_{i,B_{11}}, \ldots, \widehat{N}_{i,18} := N_{i,F_2}$. Then $\eta^\top N_i^\top N_i\eta \le \delta^\top \widehat{N}_i\delta$, where $\widehat{N}_i = [n_{i,pq}] \in \mathbb{R}_+^{18\times 18}$, $i = 1,2$, is a matrtix with elements $n_{i,pq} := \left\|\widehat{N}_{i,p}^\top \widehat{N}_{i,q}\right\|_2$, $p,q = 1,\ldots,18$. Therefore we find a perturbation bound $\delta_{X_i}$ $\le \mathrm{est}_i^{(3)}(\delta) + O(\|\delta\|^2),\ \delta \to 0$, where $\mathrm{est}_i^{(3)}(\delta) := \sqrt{\delta^\top \widehat{N}_i\delta}$.

We have the overall estimates

$$\delta_{X_i} = \mathrm{est}_i(\delta) + O(\|\delta\|^2),\ \delta \to 0,\ i = 1,2,$$

where $\mathrm{est}_i(\delta) := \min\left\{\mathrm{est}_i^{(2)}(\delta), \mathrm{est}_i^{(3)}(\delta)\right\}$. The local bounds considered in this section are continuous, first order homogeneous, non-linear functions in $\delta$. Also, for $\delta \neq 0$ these functions are real analytic.

The bounds $\mathrm{est}_i^{(k)}$ are in fact majorants for the solution of a complicated optimization problem, defining the conditioning of the problem as follows. Set $\xi_i := \mathrm{vec}\,(\delta X_i)$, and $\delta$ $:= [\delta_1, \ldots, \delta_{18}]^\top := [\delta_{A_{11}}, \ldots, \delta_{F_2}]^\top \in \mathbb{R}_+^{18}$. Then we have $\xi_i = \sum_{k=1}^{18} \widehat{N}_{i,k}\eta_k + O(\|\delta\|^2)$, $\delta \to 0$ and $\delta_{X_i} = \|\xi_i\|_2 \le K_i(\delta) + O(\|\delta\|^2),\ \delta \to 0$. Here

$$K_i(\delta) := \max\left\{\left\|\sum_{k=1}^{18} \widehat{N}_{i,k}\eta_k\right\|_2 : \|\eta_k\| \le \delta_k,\ k = 1,\ldots,18\right\}$$

is the exact upper bound for the first order term in the perturbation bound for the solution component $X_i$.

The calculation of $K_i(\delta)$ is a difficult task. Instead, one can use a bound such as $\mathrm{est}_i(\delta) \ge K_i(\delta)$.

Let $\gamma \in \mathbb{R}_+^{18}$ be a given vector. Then we may define the relative conditioning of the problem as follows.

Let $X_i \neq 0$. The quantity $\kappa_i(\gamma) := \dfrac{K_i(\gamma)}{\|X_i\|_\mathrm{F}}$ is the *relative condition number of $X_i$ with respect to $\gamma$.*

If $|||P|||$ is the generalized norm of $P$, then $\kappa_i(|||P|||)$ is the *relative norm-wise condition number* of $X_i$.

If all elements $\gamma_k$ of $\gamma$ are zero except one, equal to $\|\mathcal{E}_l\|_{\mathrm{F}}$ in the $l$-th position, then $\kappa_i(\gamma)$ is the individual relative condition number of $X_i$ with respect to perturbations in the matrix $\mathcal{E}_l$.

**Non-local perturbation analysis.** The perturbed equation $F(X+\delta X, P+\delta P) = 0$ may be rewritten as an operator equation for $\delta X$

$$\delta X = \Pi(\delta X, \delta P),\ \ \Pi = (\Pi_1, \Pi_2), \tag{8}$$

where $\Pi(Y, \delta P) := -\mathbf{M}(G_P(X,P)(\delta P) + H(Y, \delta P))$. Here $H(Y, \delta P) := (H_1(Y, \delta P_1), H_2(Y, \delta P_2))$ contains second and third order terms in $Y$ and $\delta P$.

Equation (8) comprizes two equations, namely

$$\delta X_i = \Pi_i(\delta X, \delta P_i),\ \ i = 1, 2, \tag{9}$$

where the right-hand side of (9) is defined by relations (6). Setting $\xi_i := \mathrm{vec}\,(\delta X_i) \in \mathbb{R}^{n^2}$, $i = 1, 2$, $\xi := \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in \mathbb{R}^{2n^2}$, we obtain the vector operator equation

$$\xi = \pi(\xi, \eta) \in \mathbb{R}^{2n^2}, \tag{10}$$

which is reduced to two coupled vector equations $\xi_i = \pi_i(\xi, \eta) \in \mathbb{R}^{n^2}$, $i = 1, 2$ with $\pi_i(\xi, \eta) = N_i \eta_i + \psi_i(\xi, \eta)$, where $\psi_i(\xi, \eta) := -\mathrm{vec}\left( \sum_{j=1}^{2} M_{ij} \mathrm{vec}\ \left( H_j\ \left( \mathrm{vec}^{-1}(\xi), \mathrm{vec}^{-1}(\eta_j) \right) \right) \right)$.
Next we apply the method of Lyapunov majorants and the fixed point principles of Banach and Schauder [1] for the analysis of operator equation (10) in order to find non-local perturbation bounds for $\delta_{X_i} = \|\xi_i\|_2$.

The vectorizations of the matrices $H_i(Y, \delta P_i)$ are

$$\begin{aligned}
\mathrm{vec}\,(H_1(Y, \delta P_1)) \ = \ & (I_n \otimes X_1)\mathrm{vec}\,(\delta D_{11} Y_1 + \delta E_1 Y_2) \\
& + (X_1^\top \otimes I_n)\mathrm{vec}\,(Y_1 \delta D_{11} + Y_2 \delta F_1) \\
& + (I_n \otimes X_2)\mathrm{vec}\,(\delta F_1 Y_1 + \delta D_{12} Y_2) \\
& + (X_2^\top \otimes I_n)\mathrm{vec}\,(Y_1 \delta E_1 + Y_2 \delta D_{12}) \\
& + \mathrm{vec}\,(\delta A_{11} Y_1 + \delta A_{12} Y_2) + \mathrm{vec}\,(Y_1 \delta B_{11} + Y_2 \delta B_{12}) \\
& + \mathrm{vec}\ (Y_2(F_1 + \delta F_1) Y_1 + Y_1 (E_1 + \delta E_1) Y_2) \\
& + \mathrm{vec}\ (Y_1 (D_{11} + \delta D_{11}) Y_1) + \mathrm{vec}\ (Y_2 (D_{12} + \delta D_{12}) Y_2)
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\mathrm{vec}\,(H_2(Y, \delta P_2)) \ = \ & (I_n \otimes X_1)\mathrm{vec}\,(\delta D_{21} Y_1 + \delta F_2 Y_2) \\
& + (X_1^\top \otimes I_n)\mathrm{vec}\,(Y_1 \delta D_{21} + Y_2 \delta E_2) \\
& + (I_n \otimes X_2)\mathrm{vec}\,(\delta D_{22} Y_2 + \delta E_2 Y_1) \\
& + (X_2^\top \otimes I_n)\mathrm{vec}\,(Y_2 \delta D_{22} + Y_1 \delta F_2) \\
& + \mathrm{vec}\,(\delta A_{21} Y_1 + \delta A_{22} Y_2) + \mathrm{vec}\,(Y_1 \delta B_{21} + Y_2 \delta B_{22}) \\
& + \mathrm{vec}\ (Y_2(E_2 + \delta E_2) Y_1 + Y_1 (F_2 + \delta F_2) Y_2) \\
& + \mathrm{vec}\ (Y_1 (D_{21} + \delta D_{21}) Y_1) + \mathrm{vec}\ (Y_2 (D_{22} + \delta D_{22}) Y_2)\,.
\end{aligned} \tag{12}$$

Let $\|Y_i\|_{\mathrm{F}} \le \rho_i$, $i = 1, 2$, where $\rho_i$ are non-negative constants. Then it follows from

(11), (12) that $\|\pi_i(\xi,\eta)\|_2 \le \mathrm{est}\,_i(\delta) + \sum_{j=1}^2 \|M_{ij}\mathrm{vec}\,(H_j(Y,\delta P_j))\|_2 \le h_i(\rho,\delta)$, where $\rho = [\rho_1,\rho_2]^\top \in \mathbb{R}_+^2$ and $h_i(\rho_1,\rho_2,\delta) := \mathrm{est}\,_i(\delta) + a_{i1}(\delta)\rho_1 + a_{i2}(\delta)\rho_2 + 2b_i(\delta)\rho_1\rho_2 + c_{i1}(\delta)\rho_1^2$ $+c_{i2}(\delta)\rho_2^2$, $i = 1,2$. Here

$$\begin{aligned}
a_{i1}(\delta) &:= \|M_{i1}\|_2(\delta_{A_{11}} + \delta_{B_{11}}) + \|M_{i2}\|_2(\delta_{A_{21}} + \delta_{B_{21}}) \\
&\quad + (\nu_{i11} + \nu_{i13})\delta_{D_{11}} + \nu_{i12}\delta_{F_1} + \nu_{i14}\delta_{E_1} \\
&\quad + (\nu_{i21} + \nu_{i23})\delta_{D_{21}} + \nu_{i22}\delta_{E_2} + \nu_{i24}\delta_{F_2}, \\
a_{i2}(\delta) &:= \|M_{i1}\|_2(\delta_{A_{12}} + \delta_{B_{12}}) + \|M_{i2}\|_2(\delta_{A_{22}} + \delta_{B_{22}}) \\
&\quad + (\nu_{i12} + \nu_{i14})\delta_{D_{12}} + \nu_{i11}\delta_{E_1} + \nu_{i13}\delta_{F_1} \\
&\quad + (\nu_{i22} + \nu_{i24})\delta_{D_{22}} + \nu_{i21}\delta_{F_2} + \nu_{i23}\delta_{E_2}, \\
b_i(\delta) &:= \|M_{i1}\|_2(\|F_1\|_2 + \delta_{F_1} + \|E_1\|_2 + \delta_{E_1}) \\
&\quad + \|M_{i2}\|_2(\|F_2\|_2 + \delta_{F_2} + \|E_2\|_2 + \delta_{E_2}), \\
c_{i1}(\delta) &:= \|M_{i1}\|_2(\|D_{11}\|_2 + \delta_{D_{11}}) + \|M_{i2}\|_2(\|D_{21}\|_2 + \delta_{D_{21}}), \\
c_{i2}(\delta) &:= \|M_{i1}\|_2(\|D_{12}\|_2 + \delta_{D_{12}}) + \|M_{i2}\|_2(\|D_{22}\|_2 + \delta_{D_{22}}),\ i = 1,2,
\end{aligned}$$

and $\nu_{i11} := \|M_{i1}(I_n \otimes X_1)\|_2$, $\nu_{i12} := \|M_{i1}(I_n \otimes X_2)\|_2$, $\nu_{i13} := \left\|M_{i1}(X_1^\top \otimes I_n)\right\|_2$, $\nu_{i14} := \left\|M_{i1}(X_2^\top \otimes I_n)\right\|_2$, $\nu_{i21} := \|M_{i2}(I_n \otimes X_1)\|_2$, $\nu_{i22} := \|M_{i2}(I_n \otimes X_2)\|_2$, $\nu_{i23} := \left\|M_{i2}(X_1^\top \otimes I_n)\right\|_2$, $\nu_{i24} := \left\|M_{i2}(X_2^\top \otimes I_n)\right\|_2$.

The function $h : \mathbb{R}_+^2 \times \mathbb{R}_+^{18} \to \mathbb{R}_+^2$ is a vector Lyapunov majorant for the operator equation (10).

Consider the majorant system of two scalar quadratic equations

$$\rho_i = h_i(\rho_1,\rho_2,\delta),\quad i = 1,2, \tag{13}$$

which may also be written in vector form as $\rho = h(\rho,\delta)$, where $h := [h_1,h_2]^\top$. We have $h(0,0) = 0$, $h_\rho(0,0) = 0$. Therefore, according to the theory of Lyapunov majorants, for $\delta$ sufficiently small, the system (13) has a solution $\rho = f(\delta) = [f_1(\delta), f_2(\delta)]^\top$, which is continuous, real analytic in $\delta \ne 0$ and satisfies $\rho(0) = 0$. The function $f(\cdot)$ is defined in a domain $\Omega \subset \mathbb{R}_+^{18}$ whose boundary $\partial\Omega$ may be obtained by excluding $\rho$ from the system of equations $\rho = h(\rho,\delta)$, $\det(I_2 - h_\rho(\rho,\delta)) = 0$. The second equation is equivalent to $\omega(\rho,\delta) := 1 - \varepsilon(\delta) + \alpha_1(\delta)\rho_1 + \alpha_2(\delta)\rho_2 + 2\beta(\delta)\rho_1\rho_2 + \gamma_1(\delta)\rho_1^2 + \gamma_2(\delta)\rho_2^2 = 0$, where $\varepsilon(\delta) := a_{11}(\delta) + a_{22}(\delta) - a_{11}(\delta)a_{22}(\delta) + a_{12}(\delta)a_{21}(\delta)$, $\alpha_1(\delta) := -2\ c_{11}(\delta)(1 - a_{22}(\delta)) - b_2(\delta)(1 - a_{11}(\delta)) - 2\ a_{12}(\delta)c_{21}(\delta) - b_1(\delta)a_{21}(\delta)$, $\alpha_2(\delta) := -2\ c_{22}(\delta)(1 - a_{11}(\delta)) - b_1(\delta)(1 - a_{22}(\delta)) - 2\ a_{21}(\delta)c_{12}(\delta) - b_2(\delta)a_{12}(\delta)$, $\beta(\delta) := 2(c_{11}(\delta)c_{22}(\delta) - c_{12}(\delta)c_{21}(\delta))$, $\gamma_1(\delta) := 2(b_2(\delta)c_{11}(\delta) - b_1(\delta)c_{21}(\delta))$, $\gamma_2(\delta) := 2(b_1(\delta)c_{22}(\delta) - b_2(\delta)c_{12}(\delta))$. Thus for the determination of the boundary $\partial\Omega$ of the set $\Omega$ we have a system of 3 scalar full 2-nd degree equations in $\rho_1$, $\rho_2$, whose coefficients are 2-nd degree polynomials in $\delta$. For $\delta \in \Omega$ denote by $\rho = f(\delta)$ the smallest non-negative solution of the majorant system (13). If the system (13) has not a smallest solution in $\mathbb{R}_+^2$, we can take any solution $\rho = f(\delta) \in \mathbb{R}_+^2$ such that $\omega(f(\delta),\delta) \ge 0$.

Thus the operator $\pi(\cdot,\eta)$ maps the closed convex set $\mathcal{B}_\rho = \{\xi : \|\xi_i\| \le \rho_i,\ i = 1,2\}$ into itself. Hence according to the Schauder fixed point principle there is a solution $\xi \in \mathcal{B}_\rho$ of the operator equation $\xi = \pi(\xi,\eta)$. As a result we have the non-local non-linear perturbation bounds $\delta_{X_i} \le f_i(\delta)$, $\delta \in \Omega$.

In practice it is not necessary to determine explicitly the domain $\Omega$ and the functions $f_i$. It suffices, for a given $\delta$, to solve numerically the majorant system (13) and then to check the condition $\omega(\widetilde{\rho}, \delta) \geq 0$, where $\widetilde{\rho}$ is the computed solution. This 'numerical' approach to the non-local perturbation analysis may be avoided, obtaining explicit perturbation bounds. The idea is to find a new Lyapunov majorant $k = [k_1, k_2]^\top$, such that $h(\rho, \delta) \preceq k(\rho, \delta)$ and for which the equation $\rho = k(\rho, \delta)$ has an explicit solution.

Let $k_i(\delta, \rho) := e_i + a_1\rho_1 + a_2\rho_2 + 2b\rho_1\rho_2 + c_1\rho_1^2 + c_2\rho_2^2$. It is easy to see that $k$ is a Lyapunov majorant. The solution of the majorant system $\rho = k(\rho, \delta)$ will majorize the solution of the system $\rho = h(\rho, \delta)$. We have $\rho_1 = \rho_2 + e_1 - e_2$. Using the equations $\rho_i = k_i(\rho, \delta)$ we obtain

$$\delta_{X_i} \leq \rho_i = \frac{2\left(a_j e_j + (1 - a_j)e_i + c_j(e_1 - e_2)^2\right)}{1 - a_1 - a_2 + 2(b + c_j)(e_i - e_j) + \sqrt{d_k}}, \ i = 1, 2, \tag{14}$$

where

$$\begin{aligned} d_k &= d_k(\delta) := (1 - a_1 - a_2)^2 - 4(a_1(b + c_2) + (1 - a_2)(b + c_1))e_1 \\ &\quad - 4(a_2(b + c_1) + (1 - a_1)(b + c_2))e_2 + 4(b^2 - c_1c_2)(e_1 - e_2)^2 \end{aligned}$$

and $j \neq i$. These bounds hold provided $\delta \in \Theta_k := \left\{\delta \in \mathbb{R}_+^{18} : d_k(\delta) \geq 0\right\}$.

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## СМУЩЕНИЯ В ОБЩИ КУПЛИРАНИ МАТРИЧНИ УРАВНЕНИЯ НА РИКАТИ


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Изведени са пертурбационни граници за общите алгебрични непрекъснати куплирани матрични уравнения на Рикати, възникващи в съвременната теория на управлението.