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MEASURABILITY OF SETS OF PAIRS OF SKEW NONISOTROPIC AND ISOTROPIC STRAIGHT LINES IN THE SIMPLY ISOTROPIC SPACE*

Adrijan V. Borisov, Margarita G. Spirova

We study the measurability of sets of pairs of skew nonisotropic and isotropic straight lines and the corresponding invariant densities with respect to the group of the general similitudes and some of its subgroups.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is defined (see [7]) as a projective space $\mathbb{P}_3(\mathbb{R})$ with an absolute consisting of a plane ω (the absolute figure) and two complex conjugate straight lines f_1, f_2 (the absolute plane) into ω with a (real) intersection point F (the absolute point). In homogeneous coordinates (x_0, x_1, x_2, x_3) we can take the plane $x_0 = 0$ as the plane ω, the line $x_0 = 0, x_1 + ix_2 = 0$ as the line f_1, the line $x_0 = 0, x_1 - ix_2 = 0$ as the line f_2 and the point $(0, 0, 0, 1)$ as the point F. All regular projectivities transforming the absolute figure into itself form the 8-parameter group G_8 of the general simply isotropic similitudes. Passing on to affine coordinates (x, y, z), any similitude of G_8 can be written in the form [7; p. 3]

$$\begin{aligned} x' &= c_1 + c_7(x\cos\varphi - y\sin\varphi), \\ y' &= c_2 + c_7(x\sin\varphi + y\cos\varphi), \\ z' &= c_3 + c_4x + c_5y + c_6z, \end{aligned} \tag{1}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0$ and φ are real parameters.

A straight line in $I_3^{(1)}$ is said to be (completely) isotropic if its infinite point coincides with the absolute point F; otherwise the straight line is said to be nonisotropic [7; p. 5].

We shall consider G_8 and the following its subgroups:

I. $B_7 \subset G_8 \iff c_7 = 1$. It is the group of the simply isotropic similitudes of the δ-distance [7; p. 5].

II. $S_7 \subset G_8 \iff c_6 = 1$. It is the group of the simply isotropic similitudes of the s-distance [7; p. 6].

III. $W_7 \subset G_8 \iff c_6 = c_7$. It is the group of the simply isotropic angular similitudes [7; p. 18].

IV. $G_7 \subset G_8 \iff \varphi = 0$. It is the group of the boundary simply isotropic similitudes [7; p. 8].

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V. $V_7 \subset G_8 \iff c_6 c_7^2 = 1$. It is the group of the volume preserving simply isotropic similitudes [7: p. 8].

VI. $G_6 = G_7 \cap V_7$. It is the group of the volume preserving boundary simply isotropic similitudes [7; p. 8].

VII. $B_6 = B_7 \cap G_7$. It is the group of the modular boundary motions [7; p. 9].

VIII. $B_5 = B_7 \cap S_7 \cap G_7$. It is the group of the unimodular boundary motions [7; p. 9].

We emphasize that most of the common material of the geometry of the simply isotropic space ${I_3}^{(1)}$ can be found in [7], [9] and [10].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [8], G. I. Drinfel'd and A. V. Lucenko [4], [5], [6] we study the measurability of sets of pairs of skew nonisotropic and isotropic straight lines in ${I_3}^{(1)}$ with respect to G_8 and the indicated above subgroups. Analogous problems for sets of pairs of skew nonisotropic straight lines in ${I_3}^{(1)}$ have been treated in [1] and [2].

2. Measurability with respect to G_8. Let (G, J) be a pair of skew nonisotropic and isotropic straight lines determined by the equations

$$G: \ x = \alpha z + \lambda_1, \ y = \beta z + \mu_1, \qquad |\alpha| + |\beta| \neq 0 \tag{2}$$

and

$$J: \ x = \lambda_2, \ y = \mu_2, \tag{3}$$

respectively. We can assume without loss of generality that $\alpha \neq 0$ and then we can take the Pluecker coordinates [7, p. 38–41] p_2, p_3, p_5, p_6, q_4, q_5 as the parameters of the set of pairs (G, J), where

$$p_2 = \frac{\beta}{\alpha}, \quad p_3 = \frac{1}{\alpha}, \quad p_5 = -\frac{\lambda_1}{\alpha}, \quad p_6 = \frac{\beta\lambda_1 - \alpha\mu_1}{\alpha}, \quad q_4 = \mu_2, \quad q_5 = -\lambda_2. \tag{4}$$

Under the action of (1) the pair $(G, J)(p_2, p_3, p_5, p_6, q_4, q_5)$ is transformed into the pair $(G', J')({p_2}', {p_3}', {p_5}', {p_6}', {q_4}', {q_5}')$ as follows:

$$\begin{aligned}
{p_2}' &= c_7 K\,(\sin\varphi + p_2\cos\varphi),\\
{p_3}' &= \quad K(c_4 + c_5 p_2 + c_6 p_3),\\
{p_5}' &= K\{(c_3 - c_5 p_6 + c_6 p_5)c_7\cos\varphi -\\
&\qquad [\,c_3 + c_4 p_6 + c_6(p_2 p_5 + p_3 p_6)]c_7\sin\varphi - c_1(c_4 + c_5 + c_6 p_3)\},\\
{p_6}' &= c_7 K[(c_1 p_2 - c_2)\cos\varphi + (c_1 + c_2 p_2)\sin\varphi + c_7 p_6],\\
{q_4}' &= c_7(q_4\cos\varphi - q_5\sin\varphi) + c_2,\\
{q_5}' &= c_7(q_4\sin\varphi + q_5\cos\varphi) - c_1,
\end{aligned} \tag{5}$$

where $K = [c_7(\cos\varphi - p_2\sin\varphi)]^{-1}$. The transformations (5) form the associated group $\overline{G}_8$ of G_8 [8, p. 34]. The group $\overline{G}_8$ is isomorphic to G_8 and the invariant density with respect to G_8 of the pairs of lines (G, J), if it exists, coincides with the density with respect to $\overline{G}_8$ of the points $(p_2, p_3, p_5, p_6, q_4, q_5)$ in the set of parameters.

The associated group $\overline{G}_8$ has the infinitesimal operators

$$
(6)\qquad
\begin{aligned}
&Y_1 = -p_3\frac{\partial}{\partial p_5} + p_2\frac{\partial}{\partial p_6} - \frac{\partial}{\partial q_5}, \quad Y_2 = -\frac{\partial}{\partial p_6} + \frac{\partial}{\partial q_4}, \quad Y_3 = \frac{\partial}{\partial p_5},\\
&Y_4 = -p_3\frac{\partial}{\partial p_3} + p_6\frac{\partial}{\partial p_6} + q_4\frac{\partial}{\partial q_4} + q_5\frac{\partial}{\partial q_5},\\
&Y_5 = (1+p_2^2)\tfrac{\partial}{\partial p_2} + p_2p_3\tfrac{\partial}{\partial p_3} - p_3p_6\tfrac{\partial}{\partial p_5} + p_2p_6\tfrac{\partial}{\partial p_6} - q_5\tfrac{\partial}{\partial q_4} + q_4\tfrac{\partial}{\partial q_5},\\
&Y_6 = \frac{\partial}{\partial p_3}, \quad Y_7 = p_2\frac{\partial}{\partial p_3} - p_6\frac{\partial}{\partial p_5}, \quad Y_8 = p_3\frac{\partial}{\partial p_3} + p_5\frac{\partial}{\partial p_5}
\end{aligned}
$$

and it acts transitively on the set of points $(p_2, p_3, p_5, p_6, q_4, q_5)$. It is easy to verify that Y_1, Y_2, Y_3, Y_4, Y_5 and Y_6 are arcwise unconnected but $Y_7 = -p_6Y_3 + p_2Y_6$, $Y_8 = p_5Y_3 + p_3Y_6$. Since $Y_3(p_5) + Y_6(p_3) \neq 0$, we conclude that the following statement holds:

Theorem 2.1. *A set of pairs of skew nonisotropic and isotropic straight lines is not measurable with respect to the group G_8 and it has no measurable subsets.*

3. Measurability with respect to S_7. The associated group $\overline{S}_7$ of the group S_7 has the infinitesimal operators Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 and Y_7 from (6) and it acts transitively on the set of parameters $(p_2, p_3, p_5, p_6, q_4, q_5)$. The integral invariant function $f = f(p_2, p_3, p_5, p_6, q_4, q_5)$ satisfies the so-called system of R. Deltheil [3, p. 28], [8, p. 11] $Y_1(f) = 0, Y_2(f) = 0, Y_3(f) = 0, Y_4(f) + 2f = 0,\ Y_5(f) + 4p_2f = 0, Y_6(f) = 0, Y_7(f) = 0$ and has the form $f = \dfrac{h}{(p_2q_5 + p_6 + q_4)^2(1 + p_2^2)}$, where $h = \text{const}$.

Thus we established the following

Theorem 3.1. *The set of pairs $(G, J)(p_2, p_3, p_5, p_6, q_4, q_5)$ is measurable with respect to the group S_7 and has the density*

$$
d(G, J) = \frac{1}{(p_2q_5 + p_6 + q_4)^2(1 + p_2^2)}\, dp_2 \wedge dp_3 \wedge dp_5 \wedge dp_6 \wedge dq_4 \wedge dq_5. \tag{7}
$$

Remark 3.1. We note that a nonisotropic straight line $G(p_2, p_3, p_5, p_6)$ and an isotropic straight line $J(q_4, q_5)$ are skew iff [7, p. 43] $p_2q_5 + p_6 + q_4 \neq 0$.

Differentiating (4) and substituting into (7) we obtain another expression for the density:

Corollary 3.1. *The density* (7) *for the pairs* (G, J) *can be written of the form*

$$
d(G, J) = \frac{1}{(\alpha^2 + \beta^2)[(\mu_2 - \mu_1)\alpha - (\lambda_2 - \lambda_1)\beta]^2}\, d\alpha \wedge d\beta \wedge d\lambda_1 \wedge d\mu_1 \wedge d\lambda_2 \wedge d\mu_2.
$$

4. Measurability with respect to G_6. The associated group $\overline{G}_6$ of the group G_6 has the infinitesimal operators

$$Y_1 = -p_3\frac{\partial}{\partial p_5} + p_2\frac{\partial}{\partial p_6} - \frac{\partial}{\partial q_5}, \quad Y_2 = -\frac{\partial}{\partial p_6} + \frac{\partial}{\partial q_4}, \quad Y_3 = \frac{\partial}{\partial p_5},$$

$$Y_4 = -3p_3\frac{\partial}{\partial p_3} - 2p_5\frac{\partial}{\partial p_5} + p_6\frac{\partial}{\partial p_6} + q_4\frac{\partial}{\partial q_4} + q_5\frac{\partial}{\partial q_5}, \ Y_5 = \frac{\partial}{\partial p_3}, \ Y_6 = p_2\frac{\partial}{\partial p_3} - p_6\frac{\partial}{\partial p_5}.$$

Since $\overline{G}_6$ acts intransitively on the set of points $(p_2, p_3, p_5, p_6, q_4, q_5)$, the set of pairs (G, J) is not measurable with respect to G_6. The system $Y_i(f) = 0$, $i = 1, \ldots, 6$, has the solution $f = p_2$ and it is an absolute invariant of $\overline{G}_6$.

Consider the subset of pairs (G, J) satisfying the condition

$$p_2 = h, \tag{8}$$

where h = const. The group $\overline{G}_6$ induces the group $G_6^\star$ on the subset (8) with the infinitesimal operators

$$Z_1 = -p_3\frac{\partial}{\partial p_5} + h\frac{\partial}{\partial p_6} - \frac{\partial}{\partial q_5}, \quad Z_2 = -\frac{\partial}{\partial p_6} + \frac{\partial}{\partial q_4}, \quad Z_3 = \frac{\partial}{\partial p_5},$$

$$Z_4 = -3p_3\frac{\partial}{\partial p_3} - 2p_5\frac{\partial}{\partial p_5} + p_6\frac{\partial}{\partial p_6} + q_4\frac{\partial}{\partial q_4} + q_5\frac{\partial}{\partial q_5}, \ Z_5 = \frac{\partial}{\partial p_3}, \ Z_6 = h\frac{\partial}{\partial p_3} - p_6\frac{\partial}{\partial p_5}$$

and obviously it is transitive. The Deltheil system

$$Z_1(f) = 0, \ Z_2(f) = 0, \ Z_3(f) = 0, \ Z_4(f) - 2f = 0, \ Z_5(f) = 0, \ Z_6(f) = 0$$

has the solution $f = c(hq_5 + q_4 + p_6)^2$, where c = const.

From here it follows

Theorem 4.1. *The set of pairs $(G, J)(p_2, p_3, p_5, p_6, q_4, q_5)$ is not measurable with respect to the group G_6 but it has the measurable subset* (8) *with the density*

$$d(G, J) = (hq_5 + q_4 + p_6)^2 dp_3 \wedge dp_5 \wedge dp_6 \wedge dq_4 \wedge dq_5.$$

From Theorem 4.1. and (4), by direct computation, we obtain

Corollary 4.1. *The set of pairs $(G, J)(\alpha, \beta, \lambda_1, \mu_1, \lambda_2, \mu_2)$, determined by* (2) *and* (3), *is not measurable with respect to the group G_6 but it has the measurable subset*

$$\frac{\beta}{\alpha} = h, \qquad h = \text{const}$$

with the density

$$d(G, J) = |\frac{[(\lambda_2 - \lambda_1)h - (\mu_2 - \mu_1)]^2}{\alpha^3}| \, d\alpha \wedge d\lambda_1 \wedge d\mu_1 \wedge d\lambda_2 \wedge d\mu_2.$$

5. Measurability with respect to B_7, W_7, G_7, V_7, B_6 and B_5. By arguments similar to the ones used above we study the measurability of sets of pairs (G, J) with respect to all the rest groups. We collect the results in the following table:

	a set of pairs (G,J) (Pluecker coordinates)	the density of (G,J) in parameters $\alpha,\beta,\lambda_1,\mu_1,\lambda_2,\mu_2$
B_7	it is not measurable and has no measurable subsets	
W_7	$d(G,J)=\dfrac{dp_2\wedge dp_3\wedge dp_5\wedge dp_6\wedge dq_4\wedge dq_5}{(p_2q_5+p_6+q_4)^4}$	$d(G,J)=\dfrac{d\alpha\wedge d\beta\wedge d\lambda_1\wedge d\mu_1\wedge d\lambda_2\wedge d\mu_2}{[(\mu_2-\mu_1)\alpha-(\lambda_2-\lambda_1)\beta]^4}$
G_7	it is not measurable and has no measurable subsets	
V_7	$d(G,J)=\dfrac{(p_2q_5+p_6+q_4)^2}{(1+p_2^2)^3}\times$ $\times dp_2\wedge dp_3\wedge dp_5\wedge dp_6\wedge dq_4\wedge dq_5$	$d(G,J)=\dfrac{[(\mu_2-\mu_1)\alpha-(\lambda_2-\lambda_1)\beta]^2}{(\alpha^2+\beta^2)^3}\times$ $\times d\alpha\wedge d\beta\wedge d\lambda_1\wedge d\mu_1\wedge d\lambda_2\wedge d\mu_2$
B_6	it is not measurable and has no measurable subsets	
B_5	it is not measurable but it has the measurable subset $p_2=h_1,\ p_2q_5+p_6+q_4=h_2$, $h_1,h_2=\text{const}$, with the density $d(G,J)=dp_3\wedge dp_5\wedge dq_4\wedge dq_5$	it is not measurable but it has the measurable subset $\beta=\alpha h_1,\ (\mu_2-\mu_1)\alpha-(\lambda_2-\lambda_1)\beta=\alpha h_2$, $h_1,h_2=\text{const}$, with the density $d(G,J)=\lvert\alpha^{-3}\rvert\, d\alpha\wedge d\lambda_1\wedge d\lambda_2\wedge d\mu_2$

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Adrijan Varbanov Borisov
Dept. of Descriptive Geometry
Univ. of Architecture,
Civil Eng. and Geodesy
1, Christo Smirnenski Blvd.
1046 Sofia, Bulgaria
e-mail: adribor_fgs@uacg.bg

Margarita Georgieva Spirova
Fac. of Math. and Inform.
Shumen University
"Episkop Konstantin Preslavski"
115, Alen Mak Str.
9712 Shumen, Bulgaria
e-mail: margspr@fmi.shu-bg.net

ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ КРЪСТОСАНИ НЕИЗОТРОПНА И ИЗОТРОПНА ПРАВА В ПРОСТО ИЗОТРОПНО ПРОСРТАНСТВО

Адриян В. Борисов, Маргарита Г. Спирова

В статията е изследвана измеримостта на множества от двойки кръстосани неизотропна и изотропна права в просто изотропно пространство и са получени съответните гъстоти относно групата на подобностите и някои нейни подгрупи.