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PERTURBATIONS IN THE CANONICAL EQUATIONS OF SECOND-DEGREE SURFACES*

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This article studies perturbations in the canonical equations of n-dimensional second-degree surfaces. It introduces a Hausdorff type distance between those parts of the solutions of the given and the perturbed equations that are found in a bounded and closed set. Using the introduced distance, we estimate the deviation of the sets of the real and the complex solutions of the perturbed equation from the corresponding sets of solutions of the given equation. All possible cases for a second-degree surface are covered. If Δ denotes an upper bound for the absolute value of the perturbations in the equation, then two groups of cases, in which the perturbations in the set of solutions are estimated respectively with Δ and $\sqrt{\Delta}$, are considered.

1. Introduction. The existing geometric interpretation when $n = 2$ and $n = 3$ is useful for the investigation of the impact of perturbations in second-degree equations with n unknowns, $n \geq 2$. The second-degree curves and surfaces clarify the problems comming from perturbations in the equations and help in solving those problems. The first problem when studying the impact of perturbations in the equation on the set of its solutions is the selection of suitable characteristics of the proximity between the sets of solutions of the initial equation and the perturbed one. Next comes the problem of finding conditions for which proximity of the determined kind exists.

Let us denote by $\mathbb{R}^n$ the n-dimensional real space. Further, for $n \geq 2$ the following equation is considered:

$$\sum_{k=1}^{n} \lambda_k x_k^2 + p x_n = F \tag{1}$$

with real coefficients λ_k $(k = 1, 2, \ldots, n)$, p, and F, which satisfy the relations $\sum\limits_{k=1}^{n} \lambda_k^2 \neq 0$, $\lambda_n p = 0$, $pF = 0$. For small perturbations Δ_k $(k = 1, \ldots, n)$, Δ_p, and Δ_F, for which it is true that $|\Delta_k| \leq \Delta$, $|\Delta_k| \leq \Delta$, $|\Delta_k| \leq \Delta$, the perturbed equation

$$\sum_{k=1}^{n} (\lambda_k + \Delta_k) x_k^2 + (p + \Delta_p) x_n = F + \Delta_F \tag{2}$$

is considered.

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If Q is a bounded and closed set of $\mathbb{R}^n$ and $M(x_1, \dots, x_n)$ belongs to Q, then throughout the paper the following notations are used:

$$l_1 = \min_k \{|\lambda_k| : \lambda_k \neq 0\} \tag{3}$$

$$b_1 = \max_{M \in Q} \left(\sum_{k=1}^{n} |x_k|^2 + |x_n| + 1 \right) \tag{4}$$

$$\sigma_\Delta = \sum_{k=1}^{n} \Delta_k \tilde{x}_k^2 + \Delta_p \tilde{x}_n - \Delta_F. \tag{5}$$

The minimum in (3) is taken from the coordinates with index k, for which $\lambda_k \neq 0$. Analogous notations are used further in the paper.

Let us denote by G and G_Δ the set of points $(x_1, \dots, x_n)$ with real coordinates, which satisfy respectively equations (1) and (2). If $\tilde{M}(\tilde{x}_1, \dots, \tilde{x}_n)$ and $M(x_1, \dots, x_n)$ are two points with real coordinates, then let us introduce the notations:

$$d(\tilde{M}, G) = \min_{M \in G} \max_{1 \le k \le n} |\tilde{x}_k - x_k|, \qquad d(\tilde{M}, G_\Delta) = \min_{M \in G_\Delta} \max_{1 \le k \le n} |\tilde{x}_k - x_k|.$$

Let Q be a bounded and closed set in $\mathbb{R}^n$. Then the number

$$\rho\,(Q \cap G, Q \cap G_\Delta) = \max \left\{ \max_{\tilde{M} \in Q \cap G_\Delta} d\left(\tilde{M}, G\right), \max_{\tilde{M} \in Q \cap G} d\left(\tilde{M}, G_\Delta\right) \right\} \tag{6}$$

determines the distance of Hausdorff type between the situated in the set Q parts of the sets G and G_Δ.

Let $\mathbb{C}^n$ be the $n-$dimensional complex space. Then by (6) we analogously define the number $\rho\,(Q \cap G, Q \cap G_\Delta)$ where G and G_Δ are the sets of complex solutions of the equations (1) and (2), and Q is a bounded and closed set in $\mathbb{C}^n$. In this case everywhere we denote by $|a|$ the absolute value of the complex number a.

A distance of the considered type between sets in the plane is applied in [3]. As a distance between functions a distance of the considered type is used in [2]. In [1] a distance of similar type is defined between sets of points in the complex plane.

Further, using the distance (6), we estimate the impact of small perturbations in the equation (1) on the set of real and complex solutions of the equation. The possible cases for the type of the equation (1) are distributed in two major groups: surfaces with a (finite) center and surfaces without a (finite) center.

2. Perturbations in the Canonical Equation of Surfaces with a (Finite) Center. For the surface with a (finite) center in equation (1) we have $p = 0$. Equations (1) and (2) take the form

$$\sum_{k=1}^{n} \lambda_k x_k^2 = F \tag{7}$$

$$\sum_{k=1}^{n} (\lambda_k + \Delta_k) x_k^2 + \Delta_p x_n = F + \Delta_F. \tag{8}$$

First, let G and G_Δ be the sets of points $(x_1, \dots, x_n)$ with real coordinates, which are the solutions of equations (7) and (8).

Theorem 1. *Let in equation* (7) *either the coefficient F be non-zero and the sign of at least one of the non-zero coefficients λ_k $(k = 1, \dots, n)$ coincide with the sign of F, and Q be a bounded and closed set in $\mathbb{R}^n$, or the coefficient F be zero, at least two of the coefficients λ_k $(k = 1, \dots, n)$ be non-zero and have opposite signs, and Q be a bounded and closed set in $\mathbb{R}^n$, which does not contain a point $(x_1, \dots, x_n)$, for which it is true that $\lambda_k x_k = 0$ when $k = 1, \dots, n$. Then, there exists a constant $\nu > 0$, such that inequality $\rho(Q \cap G, Q \cap G_\Delta) \leq \nu\Delta$ holds when $\Delta \to 0$.*

Proof. According to the assumptions made so far, there exists a constant $b_3 > 0$, such that $b_3 = \min\limits_{M \in Q \cap G} \max\limits_{k} \{|x_k| : \lambda_k \neq 0\}$. For the notation (3) and (4) let the positive number Δ satisfy the inequalities

$$\Delta \leq \frac{l_1}{2}, \qquad \Delta \leq \frac{l_1 b_3^2}{2b_1}. \tag{9}$$

First, let the point $\tilde{M}(\tilde{x}_1, \dots, \tilde{x}_n)$ belong to the set $Q \cap G$. The index k_0 is so chosen that the equality

$$|\tilde{x}_{k_0}| = \max_k \{|\tilde{x}_k| : \lambda_k \neq 0\} \tag{10}$$

is fulfilled. Let us introduce the number

$$\delta = \tilde{x}_{k_0}\left(-1 + \sqrt{1 - \frac{\sigma_\Delta}{(\lambda_{k_0} + \Delta_{k_0})\,\tilde{x}_{k_0}^2}}\right), \tag{11}$$

where σ_Δ is defined in (5). Since $\tilde{x}_{k_0}$ is a real number and the inequalities (9) hold, then we have:

$$\left|\frac{\sigma_\Delta}{(\lambda_{k_0} + \Delta_{k_0})\,\tilde{x}_{k_0}^2}\right| \leq \frac{\Delta b_1}{\left(l_1 - \frac{l_1}{2}\right) b_3^2} = 2\frac{\Delta b_1}{l_1 b_3^2} \leq 1.$$

Therefore, the number under the root in (11) is non-negative, and δ is a real number.

We shall prove that the point $\tilde{M}_\Delta(\tilde{x}_1, \dots, \tilde{x}_{k_0-1}, \tilde{x}_{k_0} + \delta, \tilde{x}_{k_0+1}, \dots, \tilde{x}_n)$ belongs to the set G_Δ. After substituting its coordinates in the left side of equation (8), we obtain

$$(\lambda_{k_0} + \Delta_{k_0})\left(\tilde{x}_{k_0}^2 - \frac{\sigma_\Delta}{\lambda_{k_0} + \Delta_{k_0}}\right) + \sum_{k \neq k_0} (\lambda_k + \Delta_k)\,\tilde{x}_k^2 + \Delta_p \tilde{x}_n = \sum_{k=1}^{n} (\lambda_k + \Delta_k)\,\tilde{x}_k^2 -$$

$$- \left(\sum_{k=1}^{n} \Delta_k \tilde{x}_k^2 + \Delta_p \tilde{x}_n - \Delta_F\right) + \Delta_p \tilde{x}_n = \sum_{k=1}^{n} \lambda_k \tilde{x}_k^2 + \Delta_F = F + \Delta_F.$$

In the last equality we use the fact that the point $\tilde{M}$ is from G. Therefore, really $\tilde{M}_\Delta \in G_\Delta$.

From (11) we obtain

$$\delta = \tilde{x}_{k_0}\left(1 - \frac{\sigma_\Delta}{(\lambda_{k_0} + \Delta_{k_0})\,\tilde{x}_{k_0}^2} - 1\right)\left(1 + \sqrt{1 - \frac{\sigma_\Delta}{(\lambda_{k_0} + \Delta_{k_0})\,\tilde{x}_{k_0}^2}}\right)^{-1}.$$

Since, estimating $|\delta|$, the root can be omitted, we can see that

$$|\delta| \leq \frac{2\Delta b_1}{l_1 b_3} = \nu\Delta, \tag{12}$$

where we assume $\nu = \dfrac{2b_1}{l_1 b_3}$. Then from (12) it follows that

$$d\left(\tilde{M}, G_\Delta\right) \leq |\tilde{x}_{k_0} - (\tilde{x}_{k_0} + \delta)| = |\delta| \leq \nu\Delta. \tag{13}$$

Further, let $\tilde{M}(\tilde{x}_1, \ldots, \tilde{x}_n)$ be a point from the set $Q \cap G_\Delta$. The index k_0 is chosen again from the relation (10). Using the assumption (5), let us introduce the number

$$\delta = \tilde{x}_{k_0}\left(-1 + \sqrt{1 + \frac{\sigma\Delta}{\lambda_{k_0}\tilde{x}_{k_0}^2}}\right). \tag{14}$$

As previously, we check that there is a non-negative number under the root in (14) and the point $\tilde{M}_\Delta(\tilde{x}_1, \ldots, \tilde{x}_{k_0-1}, \tilde{x}_{k_0} + \delta, \tilde{x}_{k_0+1}, \ldots, \tilde{x}_n)$ belongs to the set G. The following estimation is valid:

$$|\delta| \leq \frac{\Delta b_1}{l_1 b_3} = \frac{\nu}{2}\Delta.$$

Therefore it is true that

$$d\left(\tilde{M}, G\right) \leq |\tilde{x}_{k_0} - (\tilde{x}_{k_0} + \delta)| = |\delta| \leq \nu\Delta. \tag{15}$$

From (13) and (15) it follows that when Δ satisfies the inequalities (9), then $\rho(Q \cap G, Q \cap G_\Delta) \leq \nu\Delta$. This completes the proof of the theorem.

Theorem 2. *Let the coefficient F in the equation* (7) *be zero and at least two of the coefficients λ_k $(k = 1, \ldots, n)$ be non-zero and have opposite signs, and Q be a bounded and closed set in $\mathbb{R}^n$, which contains a point $(x_1, \ldots, x_n)$, for which it is true that $\lambda_k x_k = 0$ when $k = 1, \ldots, n$. Then there exists a positive number ν, such that $\rho(Q \cap G, Q \cap G_\Delta) \leq \nu\sqrt{\Delta}$ when $\Delta \to 0$.*

Theorems 1 and 2 exhaust all possible cases of real surfaces with a center determined by equation (7). When $F \neq 0$ if the signs of all non-zero coefficients λ_k $(k = 1, \ldots, n)$ are different from the sign of F, then for small perturbations of the coefficients of the equation it is possible the perturbed equation not to have real solutions. When $F = 0$ and if the signs of all non-zero coefficients λ_k $(k = 1, \ldots, n)$ are the same, then it is also possible that the perturbed equation not have to real solutions. These cases are covered in the following theorems, in which we evaluate the impact of perturbations on the set of complex solutions of equation (7). These theorems are presented without proofs.

Theorem 3. *Let G and G_Δ be the sets of the complex solutions of the equations* (7) *and* (8). *Then for each bounded and closed set Q from the n-dimensional complex space $\mathbb{C}^n$ there exists a positive constant ν, such that $\rho(Q \cap G, Q \cap G_\Delta) \leq \nu\sqrt{\Delta}$ for $\Delta \to 0$.*

The next theorem specifies this result.

Theorem 4. *Let G and G_Δ be the sets of the complex solutions of the equations* (7) *and* (8). *In addition, let Qbe a bounded and closed set from the n-dimensional complex space $\mathbb{C}^n$, such that the set $Q \cap G$ does not contain a point $(x_1, \ldots, x_n)$ for which it is true that $\lambda_k x_k = 0$ for $k = 1, \ldots, n$. Then there exists a constant $\nu > 0$, such that the following inequality holds*

$$\rho(Q \cap G, Q \cap G_\Delta) \leq \nu\Delta.$$

when $\Delta \to 0$.

3. Perturbations in the Canonical Equations of a Surface without a (Finite) Center. For a surface without a (finite) center in equation (1) it is true that $p \neq 0$ and $\lambda_n = 0$, $F = 0$. Then the equation has the form

$$\sum_{k=1}^{n-1} \lambda_k x_k^2 + p x_n = 0. \tag{16}$$

In this case equation (2) is

$$\sum_{k=1}^{n-1} (\lambda_k + \Delta_k) x_k^2 + \Delta_n x_n^2 + (p + \Delta_p) x_n = \Delta_F. \tag{17}$$

Again, let first G and G_Δ be sets of points with real coordinates which are solutions respectively of equations (16) and (17). Then the following theorems holds:

Theorem 5. *For every bounded and closed set Q from the n-dimensional real space $\mathbb{R}^n$ there exists a constant $\nu > 0$, such that the following inequality is true:*

$$\rho\,(Q \cap G, Q \cap G_\Delta) \leq \nu\Delta$$

when $\Delta \to 0$.

Theorem 6. *If G and G_Δ are the sets of points $(x_1, \ldots, x_n)$ with complex coordinates, which satisfy respectively equations* (16) *and* (17)*, then for each bounded and closed set Q of the n-dimensional space $\mathbb{C}^n$ there exists a constant $\nu > 0$, such that the following inequality holds:*

$$\rho\,(Q \cap G, Q \cap G_\Delta) \leq \nu\Delta$$

when $\Delta \to 0$.

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СМУЩЕНИЯ В КАНОНИЧНИТЕ УРАВНЕНИЯ НА ПОВЪРХНИНИ ОТ ВТОРА СТЕПЕН

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В работата се изучават смущенията в каноничните уравнения на n-мерните повърхнини от втора степен. Въвежда се разстояние от хаусдорфов тип между частите от решенията на даденото и смутеното уравнение, които се намират в ограничено и затворено множество. С помощта на въведеното разстояние се оценява отклонението на множеството от реалните и комплексните решения на смутеното уравнение от съответното множество на даденото уравнение. Разгледани са всички възможни случаи на повърхнина от втора степен. Ако Δ е горна граница за абсолютната стойност на смущенията в уравненията, то са определени два случая, в които смущенията в множеството от решенията се оценява съответно с Δ и $\sqrt{\Delta}$.