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ON THE STRUCTURE OF THE EFFICIENT SET*

Zdravko D. Slavov, Christina S. Evans

The paper presents an efficient set of decision-making system with convex and compact set of alternatives and finite set of multiple criteria. It is proved that the efficient set is nonempty, path-connected and compact, if the objective functions are continuous, concave and strictly quasi-concave.

1. Introduction. Let a decision-making system (X, C) be given. In this system, $X \subset R^m$ is a set of alternatives, $m \geq 1$, $|X| > 1$ and C is a finite set of multiple criteria, $|C| = n \geq 2$. Let each criterion $c_i \in C$ be defined by the continuous preference $\succeq_i$, which is a binary relation on X. It is known that each continuous preference $\succeq_i$ can be represented by a continuous objective (or utility) function $u_i : X \to R$ [2].

In this paper, let the utility functions $\{u_i\}_{i=1}^n$ be continuous on the convex and compact set X. For each $x \in X$ and $i \in [1; n]$ let denote $R_i(x) = \{y \in X : u_i(y) \geq u_i(x)\}$. It is easy to prove that the sets $\{R_i(x)\}_{i=1}^n$ are compact subset of X.

Definition of an efficient set: an alternative $x \in X$ is called efficient alternative iff there does not exist an alternative $y \in X$ such that $u_i(y) \geq u_i(x)$ for all $i \in [1; n]$ and $u_k(y) > u_k(x)$ for some $k \in [1; n]$. We denote the set of the efficient alternatives by E and it is called an efficient set.

A function u_i is concave iff $x, y \in X$ and $t \in [0; 1]$, then $u_i(tx + (1-t)y) \geq tu_i(x)$ $+(1-t)u_i(y)$ and a function u_i is strictly quasi-concave iff $x, y \in X$ and $t \in (0; 1)$, then $u_i(tx + (1-t)y) > \min(u_i(x), u_i(y))$.

In this paper, let the functions $\{u_i\}_{i=1}^n$ be concave and strictly quasi-concave on X.

2. Main results. Let denote a function $f : X \to R$, $f(x) = \sum_{i=1}^n u_i(x)$ for all $x \in X$. It is easy to show that the function f is continuous and concave on X. It is known that $E \subset \operatorname{Arg\,max}(f, X)$ and $|E| \geq 1$ [5].

Let denote a function $U : X \to R^n$, $U(x) = (u_1(x), u_2(x), ..., u_n(x))$ for all $x \in X$. It is easy to show that the function U is continuous and concave on X.

Let analyze the convex sets $\bigcap_{i=1}^n \operatorname{Arg\,max}(u_i, X)$. There are two cases:

If the set $\bigcap_{i=1}^n \operatorname{Arg\,max}(u_i, X)$ is nonempty, then a function U has a maximum on X at $x_0 \in \bigcap_{i=1}^n \operatorname{Arg\,max}(u_i, X)$. In this case, there are $E = \bigcap_{i=1}^n \operatorname{Arg\,max}(u_i, X)$ and $|E| \geq 1$.

If the set $\bigcap_{i=1}^n \operatorname{Arg\,max}(u_i, X)$ is empty, then a function U has not a maximum on X. In this case, we must only search for the efficient alternatives. It is easy to prove that if $|E| = 1$, then $\bigcap_{i=1}^n \operatorname{Arg\,max}(u_i, X)$ is nonempty. As a result, we obtain $|E| \geq 2$.

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Let denote a point-to-set mapping $\rho : X \to 2^X$ such that $\rho(x) = \{y \in X : y \in \bigcap_{i=1}^{n} R_i(x)\}$ for all $x \in X$. It is easy to show that the set $\rho(x)$ is a nonempty, convex and compact set for all $x \in X$ and there is $x \in \rho(x)$.

Theorem 1. *If $x \in X$, then $|\operatorname{Arg\,max}(f, \rho(x))| = 1$ and $\operatorname{Arg\,max}(f, \rho(x)) \subset E$.*

Proof. Clearly, there is $|\operatorname{Arg\,max}(f, \rho(x))| \geq 1$. Let choose $y_1, y_2 \in \operatorname{Arg\,max}(f, \rho(x))$, $y_1 \neq y_2$, $t \in [0; 1]$ and $z = ty_1 + (1-t)y_2$. It is known that the set $\operatorname{Arg\,max}(f, \rho(x))$ is convex, therefore there is $z \in \operatorname{Arg\,max}(f, \rho(x))$. Thus, we obtain $f(z) = f(y_1) = f(y_2)$.

For each $i \in [1; n]$ we have $u_i(z) \geq tu_i(y_1) + (1-t)u_i(y_2)$. By using this result we derive that $f(z) \geq tf(y_1) + (1-t)f(y_2) = f(y_1)$. Since $f(z) = f(y_1)$ implies $u_i(z) = tu_i(y_1) + (1-t)u_i(y_2)$ for all $i \in [1; n]$ and for all $t \in [0; 1]$. As a result, we have that $u_i(z) = u_i(y_2) + t(u_i(y_1) - u_i(y_2))$ for all $t \in [0; 1]$, therefore we find that $u_i(y_1) = u_i(y_2)$ for all $i \in [1; n]$.

Let choose $t \in (0; 1)$ and $i \in [1; n]$. It is known that the function u_i is strictly quasi-concave, therefore we obtain $u_i(z) > \min(u_i(y_1), u_i(y_2)) = u_i(y_1)$. But $u_i(z) \geq tu_i(y_1) + (1-t)u_i(y_2)$ and by using this result we have that $f(z) > tf(y_1) + (1-t)f(y_2) = f(y_1)$. This lead to a contradiction, and thus we derive $|\operatorname{Arg\,max}(f, \rho(x))| = 1$.

Let choose $y \in \operatorname{Arg\,max}(f, \rho(x))$ and assume that $y \notin E$. From condition $y \notin E$ it follows that there exists $z \in X$ such that $u_i(z) \geq u_i(y)$ for all $i \in [1; n]$ and $u_k(z) > u_k(y)$ for some $k \in [1; n]$. As a result we have that $z \in \rho(x)$ and $f(z) > f(y)$. This lead to a contradiction, therefore we derive $y \in E$, see also [5, Theorem 5]. The theorem is proved. $\square$

Corollary 1. *If $x \in X$, then $x \in E$ is equivalent to $\{x\} = \rho(x)$.*

Proof. Let $x \in E$ and assume that $\{x\} \neq \rho(x)$. From $x \in \rho(x)$ and $\{x\} \neq \rho(x)$ it follows that there exists $y \in \rho(x) \backslash \{x\}$ such that $u_i(y) \geq u_i(x)$ for all $i \in [1; n]$. Let choose $t \in (0; 1)$ and $z = tx + (1-t)y$, therefore $z \in \rho(x)$. Since $x \neq y$ implies $u_i(z) > u_i(x)$ for all $i \in [1; n]$, which contradicts condition $x \in E$ then we obtain $\{x\} = \rho(x)$.

Conversely, let $\{x\} = \rho(x)$ and assume that $x \notin E$. From condition $x \notin E$ it follows that there exists $y \in X$ such that $u_i(y) \geq u_i(x)$ for all $i \in [1; n]$ and $u_k(y) > u_k(x)$ for some $k \in [1; n]$. Thus we have that $y \in \rho(x)$ and $x \neq y$, which contradicts condition $\{x\} = \rho(x)$, therefore we obtain $x \in E$. The corollary is proved. $\square$

Let denote a function $\varphi : X \to E$ such that $\varphi(x) \in \operatorname{Arg\,max}(f, \rho(x))$ for all $x \in X$.

Corollary 2. $\varphi(X) = E$.

Proof. Clearly, from $E \subset X$ and Corollary 1 it follows that $\varphi(E) = E$. Then we obtain $\varphi(X) = E$. The corollary is proved. $\square$

Let first consider the point-to-set mapping ρ. It is easy to show that it is compact-valued mapping.

Lemma 1. *If $\{x_k\}_{k=1}^{\infty}$,$\{y_k\}_{k=1}^{\infty} \subset X$ are pair of sequences such that $\lim\limits_{k\to\infty} x_k = x_0 \in X$ and $y_k \in \rho(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{y_k\}_{k=1}^{\infty}$ whose limit belongs to $\rho(x_0)$.*

Proof. Since $y_k \in \rho(x_k)$ for all $k \in N$ implies $u_i(y_k) \geq u_i(x_k)$ for all $k \in N$ and all $i \in [1; n]$. From $\{y_k\}_{k=1}^{\infty} \subset X$ it follows that there exists a convergent sequence $\{y'_k\}_{k=1}^{\infty} \subset \{y_k\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} y'_k = y_0 \in X$, $\{x'_k\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$, $\lim\limits_{k\to\infty} x'_k = x_0$ and $y'_k \in \rho(x'_k)$. Thus we have that $u_i(y'_k) \geq u_i(x'_k)$ for all $k \in N$ and for all $i \in [1; n]$. Taking the limit as $k \to \infty$ we obtain $u_i(y_0) \geq u_i(x_0)$ for all $i \in [1;n]$. As a result there is $y_o \in \rho(x_0)$. The lemma is proved. $\square$

Lemma 2. *If* $\{x_k\}_{k=1}^{\infty} \subset X$ *is a convergent sequence to* $x_0 \in X$ *and* $y_0 \in \rho(x_0)$*, then there exists a sequence* $\{y_k\}_{k=1}^{\infty} \subset X$ *such that* $y_k \in \rho(x_k)$ *for all* $k \in N$ *and* $\lim\limits_{k\to\infty} y_k = y_0$.

Proof. Let denote the distance between y_0 and $\rho(x_k)$ by $d_k = \inf\{d(y_0, x) : x \in \rho(x_k)\}$. Since $\rho(x_k)$ is a nonempty, convex and compact set, it follows that:
if $y_0 \in \rho(x_k)$, then $d_k = 0$ and let $y_k = y_0$;
if $y_0 \notin \rho(x_k)$, then $d_k > 0$ and there exists a unique $y_k \in \rho(x_k)$ such that $d_k = d(y_0, y_k)$.

Thus we obtain a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \rho(x_k)$ for all $k \in N$. Clearly, since $\lim\limits_{k\to\infty} x_k = x_0$ implies that the sequence $\{d_k\}_{k=1}^{\infty}$ is convergent and $\lim\limits_{k\to\infty} d_k = 0$. Thus we obtain $\lim\limits_{k\to\infty} y_k = y_0$. The lemma is proved. $\square$

Theorem 2. *The point-to-set mapping* ρ *is continuos on* X.

Proof. From Lemma 1 it follows that the point-to-set mapping ρ is upper semi-continuous of X. From Lemma 2 it follows that the point-to-set mapping ρ is lower semi-continuous of X [1, 4]. Thus we obtain that the point-to-set mapping ρ is continuous of X. The theorem is proved.

Maximum Theorem [1] [3, Theorem 6.5]**.** *"Let* X *be a topological space. If* F *is a continuous, real-valued function of* X *and* B *is a continuos compact-valued point-to-set mapping from* Y *to subsets of* X*, then the point-to-set mapping* γ *defined by* $\gamma(y) = \{x \in B(y) : F(x) \geq F(x')$ *for all* $x' \in B(y)\}$ *is upper semi-continuous and compact-valued, and the functions* f *defined by* $f(y) = F(\gamma(y))$ *is a continuous function".*

Next, let us consider the function φ.

Theorem 3. *The function* φ *is continuos on* X.

Proof. From Theorem 2 and the Maximum Theorem it follows that the function φ is continuos on X, see also [4]. The theorem is proved. $\square$

Theorem 4. *The set* E *is nonempty, path-connected and compact.*

Proof. It is known that every continuous image of a nonempty, path-connected and compact set is a nonempty, path-connected and compact set [1, 4]. From Theorem 3 and Corollary 2 it follows that the set E is nonempty, path-connected and compact. The theorem is proved. $\square$

Remark 1. It is known that path-connectedness implies connectedness [4], therefore the set E is connected. In [4, Example 1.28 and Remark 1.74], there is an example where it is seen that there exists a connected set that is not path-connected.

Remark 2. If the set $\bigcap_{i=1}^{n} \operatorname{Arg\,max}(u_i, X)$ is nonempty, then $E = \bigcap_{i=1}^{n} \operatorname{Arg\,max}$ (u_i, X) and $|E| \geq 1$. From Theorem 1 it follows that $|E| = 1$.

Remark 3. If the set $\bigcap_{i=1}^{n} \operatorname{Arg\,max}(u_i, X)$ is empty, then $|E| \geq 2$. From Theorem 4 it follows that the set E is infinite and uncountable.

REFERENCES

[1] J. Green, W. Heller. Mathematical analysis and convexity with applications to economics, *Handbook of mathematical economics*, **I**, North-Holland Publishing Company, 1981.

[2] G. Debreu. Theory of Value, John Wiley and Sons, 1959.
[3] E. Klein. Mathematical methods in theoretical economics, Academic Press, 1973.
[4] A. Mukherjea, K. Pothoven. Real and Functional Analysis, Plenum Press, 1978.
[5] Z. Slavov. Weak and strong optimality and increasing concave functions, *Applied Mathematics and Computation*, **135**, (2003), 517–529.

Zdravko Slavov
Department of Mathematics
Varna Free University
Varna, Bulgaria
e-mail: slavovibz@yahoo.com

Christina Evans
Department of Mathematics
Norwich University, VT USA
e-mail: mrsevansc@yahoo.com

ВЪРХУ СТРУКТУРАТА НА ЕФЕКТИВНОТО МНОЖЕСТВО

Здравко Д. Славов, Христина С. Еванс

В статията се представя ефективно множество в система вземаща решение при изпъкнало и компактно множество от алтернативи и крайно множество от критерии. Доказва се, че ефективното множество е непразно, линейно свързано и компактно, ако целевите функции са непрекъснати, вдлъбнати и строго квазивдлъбнати.