# PRIMITIVES OF CERTAIN CLASSES OF FUNCTIONS 

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In this paper we deal with some problems connected with the teaching of the subject "definite integral in the sense of Newton" in the technical universities. To do this we need a good supply of functions integrable in the sense of Newton. We show that each piecewise monotonic function has a generalized primitive. The exposition of the material requires a minimum of preliminary concepts and does not refer to indirect assumptions.

1. Preliminaries. The idea to use the definite integral in the sense of Newton (DISN) is appropriate for teaching integrals in technical universities [1].

Let a function $f:[a, b] \rightarrow \mathbf{R}$ be given.
Definition 1.1. The continuous function $F:[a, b] \rightarrow \mathbf{R}$ is said to be a generalized primitive of $f$ if $F^{\prime}(x)=f(x)$ for all $x \in[a, b] \backslash X$, where the set $X \subset[a, b]$ is not more than countable.

Of course, the case $X=\emptyset$ is also included and here $F$ is called simply a primitive.
For a function $f$ having a generalized primitive $F$ the DISN is defined as

$$
\int_{a}^{b} f=F(b)-F(a)
$$

The use of this concept allows to avoid convergences of the type of generalized sequences (Darboux sums in particular). Of course, the natural question here is: which functions have generalized primitives? As it is well known, among those are the linearlike functions which, by definition, are the uniform limits of step-wise functions [2]. In [2] it is shown that linear-like are the functions discontinuous on a set which is not more than countable. To prove these facts is not a trivial task and this is an obstacle to the idea of using DISN when teaching integrals in technical universities.

As it was mentioned in [1], there is a compromise - it can easily be shown that each analytic function has a primitive and hence each piecewise analytic function has a generalized primitive. The other prossibility discussed in [1] is to explain (eventually at an idea level) that each continuous function has a primitive and hence that each piecewise continuous function has a generalized primitive. However, form the viewpoint of the lecturer, this approach is hardly better than the standard scholastics of 19-th century.

The above remarks reveal the necessity of a more precise analysis of the problem
Finally, what to teach in the technical universities in the field of integration?

Obviously, in order to teach Integral Calculus in any field of Natural Sciences it is necessary for the students to be convinced that the functions used in this field do have primitives. On the other hand it is good to see what can be done here by minimum means. It is the aim of this paper to find a compromise between the next two conflicting points of view:
(i) The complete simplification in teaching mathematics (asymptotically reducing it to a collection of receipts) must be avoided even in the framework of the very modest requirements of most technical universities, and
(ii) It is not good (and often not possible) to teach mathematics for its own sake in technical universities.
2. Main results. Following the argument (ii) given above, in this section we shall prove that each monotone function has a generalized primitive. In doing so we suppose that the students already have some knowledge about the concepts of limit of a sequence and continuity and differentiability of a function.

For definiteness we consider monotone functions $f$ that are increasing. We also suppose that $f$ maps the interval $[0,1]$ into itself. Indeed, if initially $f:[a, b] \rightarrow[c, d]$ then we can define the monotone function $g:[0,1] \rightarrow[0,1]$ from

$$
g(t):=\frac{f(a+(b-a) t)-c}{d-c} .
$$

Thus we are going to prove that each increasing function $f:[0,1] \rightarrow[0,1]$ has a primitive.
We first recall the concept of the entire part $[x]$ of a given number $x \geq 0$ : this is the greatest integer that is less than or equal to $x$. Next we prove several useful lemmas.

Lemma 2.1. $2[x] \leq[2 x]$.
Proof. It is sufficient to note that $[x] \leq x$ and hence the integer $2[x]$ is not greater than $2 x$. Now it follows from the definition that $2[x] \leq[2 x]$.

Lemma 2.2. For arbitrary $x \in \mathbf{R}$ it holds

$$
\lim _{n \rightarrow \infty} \frac{2^{n} x}{2^{n}}=x
$$

Proof. The proof follows again from the definition of the entire part. We have $\left[2^{n} x\right] \leq 2^{n} x<\left[2^{n} x\right]+1$. Dividing by $2^{n}$, we get $2^{-n}\left[2^{n} x\right] \leq x$ and $x<2^{-n}\left[2^{n} x\right]+2^{-1}$. Hence

$$
x-\frac{1}{2^{n}}<\frac{\left[2^{n} x\right]}{2^{n}} \leq x
$$

Passing to the limit $n \rightarrow \infty$ we obtain the required result.
Of course, the assertion of Lemma 2.2 can be written also as $\lim _{N \rightarrow \infty}[N x] / N=x$.
If $x \in[0,1]$ and $n$ is an integer, we set $x_{n}:=\left[2^{n} x\right]-1$. Let $f$ be a monotone function. For a fixed $x \in[0,1]$ we define the sequence $\left\{\mu_{n}\right\}$ by

$$
\begin{equation*}
\mu_{n}=\mu_{n}(f, x):=\frac{1}{2^{n}} \sum_{k=0}^{x_{n}} f\left(\frac{k}{2^{n}}\right)=\frac{M_{n}}{2^{n}} \tag{1}
\end{equation*}
$$

where

$$
M_{n}=M_{n}(f, x):=\sum_{k=0}^{x_{n}} f\left(\frac{k}{2^{n}}\right) .
$$

Lemma 2.3. The sequence $\left\{\mu_{n}\right\}$ is convergent.
Proof. We note first that $\left\{\mu_{n}\right\}$ is bounded. Indeed, the sum (1) contains no more than $2^{n}$ summands and each of them is less or equal to 1 . Hence $\mu_{n} \leq 1$ for each $n$.

Next we shall show that the sequence $\left\{\mu_{n}\right\}$ is increasing. First we shall rpove that $x_{n+1}>2 x_{n}$. Indeed, $2 x_{n}=2\left[2^{n} x\right]-2$ and Lemma 2.1 gives $2\left[2^{n} x\right] \leq\left[2 \cdot 2^{n} x\right]=\left[2^{n+1} x\right]$. Hence

$$
2 x_{n}=2\left[2^{n} x\right]-2 \leq\left[2^{n+1} x\right]-2=x_{n+1}-1 .
$$

To show that $\mu_{n+1} \geq \mu_{n}$ we consider the difference $\mu_{n+1}-\mu_{n}$ in the form

$$
\mu_{n+1}-\mu_{n}=\frac{1}{2^{n+1}} \sum_{p=0}^{x_{n+1}} f\left(\frac{p}{2^{n+1}}\right)-\frac{2}{2^{n+1}} \sum_{k=0}^{x_{n}} f\left(\frac{k}{2^{n}}\right)=\frac{M_{n+1}-2 M_{n}}{2^{n+1}} .
$$

To each summand $f\left(\frac{k}{2^{n}}\right)$ of $M_{n}$ we put into correspondence the summands $f\left(\frac{2 k}{2^{n+1}}\right)=$ $f\left(\frac{k}{2^{n}}\right)$ and $f\left(\frac{2 k+1}{2^{n+1}}\right)$ of $M_{n+1}$, which are obtained for $p=2 k$ and $p=2 k+1$. This is possible since the maximum value of $k$ is $x_{n}$, of $p$ is $x_{n+1}$, and, as shown above, we have $x_{n+1} \geq 2 x_{n}+1$.

After grouping the summands in $M_{n+1}-2 M_{n}$ in pairs, we obtain

$$
M_{n+1}-M_{n} \geq \sum_{k=0}^{x_{n}}\left(f\left(\frac{2 k+1}{2^{n+1}}\right)-f\left(\frac{2 k}{2^{n+1}}\right)\right) \geq 0
$$

because $f$ is increasing, i.e. $f\left(2 k+1 / 2^{n+1}\right) \geq f\left(2 k / 2^{n+1}\right)$ for all $k$.
Therefore for $x \in[0,1]$ fixed there exists the limit

$$
\begin{equation*}
F(x):=\lim n \rightarrow \infty \mu_{n}(f, x) \tag{2}
\end{equation*}
$$

and, hence, it determines the desired primitive $F$. Furthermore, it is possible to study the properties of $F$ since the knots of the division of the interval $[0, x]$ are "universal" in the sense that they do not depend on the particular $x$ and $f$.

Lemma 2.4. Let $F$ be the primitive defined by (2). Then for each $x, y \in[0,1]$ with $x<y$ the inequalities

$$
f(x)(y-x) \leq F(y)-F(x) \leq f(y)(y-x)
$$

hold.
Proof. We have

$$
M_{n}(f, y)-M_{n}(f, x)=\sum_{k=0}^{y_{n}} f\left(\frac{k}{2^{n}}\right)-\sum_{k=0}^{x_{n}} f\left(\frac{k}{2^{n}}\right)=\sum_{k=x_{n}+1}^{y_{n}} f\left(\frac{k}{2^{n}}\right)
$$

This sum contains $y_{n}-x_{n}$ summands. Each summand $f\left(\frac{k}{2^{n}}\right)$ for $k \geq\left[2^{n} x\right]+1$ is not less than $f(x)$ since $\left[2^{n} x\right] \leq 2^{n} x<\left[2^{n} x\right]+1$ and for each $k$ the inequality $f\left(\frac{k}{2^{n}}\right) \leq f(y)$ is valid.

Thus for each $n$ we have the chain of inequalities

$$
f(x)\left(y_{n}-x_{n}-1\right) \leq M_{n}(f, y)-M_{n}(f, x) \leq f(y)\left(y_{n}-x_{n}\right) .
$$

Dividing by $2^{n}$ we get

$$
f(x)\left(\frac{\left[2^{n} y\right]}{2^{n}}-\frac{\left[2^{n} x\right]}{2^{n}}-\frac{1}{2^{n}}\right) \leq \mu_{n}(f, y)-\mu_{n}(f, x) \leq f(y)\left(\frac{\left[2^{n} y\right]}{2^{n}}-\frac{\left[2^{n} x\right]}{2^{n}}\right)
$$

Using lemmas 2.2 and 2.3 and setting $n \rightarrow \infty$ the proof is complete.
Thus we have an elementary proof of the fact that a monotone continuous function has a primitive.

Corollary 2.1. If the monotone function $f$ is continuous at $x$, then $F$ is differentiable at $x$ and $F^{\prime}(x)=f(x)$.

Proof. The proof follows directly from Lemma 2.4.
This result is sufficient for most applications that are taught in technical universities. Indeed, the functions which are of interest in applications are usually both piecewise monotone and piecewise continuous.

Combining Corollary 2.1 with the definition of a generalized primitive we can summarize the above considerations as

Corollary 2.2. Each piecewise monotone and piecewise continuous function has a generalized primitive.

It is important to note that in results of this type it is not even necessary to suppose the continuity of $f$. It is well known that each monotone function is continuous everywhere with the possible exception of countably many points. When this fact is realized, we have

Corollary 2.3. Each piecewise monotone function has a generalized primitive.
It is hard to imagine that a function of any technical importance may not be piece-wise monotone.

It may well happen, however, that in certain technical university (or other high school) the problems connected with the concept of cardinality are not commented at all. As it is shown below, even in this case we can save our construction.

Definition 2.1. Let $f:[a, b] \rightarrow \mathbf{R}$ be a monotone function. The function $\Phi:[a, b] \rightarrow$ $\mathbf{R}$ is said to be a generalized primitive of $f$ if the inequalities

$$
\begin{equation*}
f(x)(y-x) \leq \Phi(y)-\Phi(x) \leq f(y)(y-x) \tag{3}
\end{equation*}
$$

are fulfilled for all $x, y \in[a, b]$.
The reader should not be confused with the standard Definition 1.1 of a generalized primitive since in Definition 2.1 we refer to monotone functions only.

Obviously each primitive $\Phi$ of $f$ in the sense of Definition 2.1 is continuous on $[a, b]$ and differentiable at the points $x$ of continuity of $f$. In these points $\Phi^{\prime}(x)=f(x)$.

Definition 2.1 allows to develop Integral Calculus for piecewise monotone functions. This is possible since for generalized primitives in the sense of Definition 2.1 the main theorem of Integral Calculus remains valid as shown below.

Theorem 2.1. Let $\Phi$ and $\Psi$ be generalized primitives of the monotone function $f$ in the interval $[a, b]$. Then their difference is a constant.

Proof. Suppose again that $[a, b]=[0,1]$. Let $\Psi$ be a generalized primitive of $f$ in the sense of Definition 2.1, and $\Phi=F$ be the function $F$ constructed in Lemma 2.3. For
a fixed $x \in[0,1]$ and for an arbitrary positive integer $n$ consider the expression

$$
\Delta_{n}=\Psi\left(\frac{x_{n}}{2^{n}}\right)-\Psi(0)
$$

We have

$$
\begin{aligned}
\Delta_{n}= & \Psi\left(\frac{x_{n}}{2^{n}}\right)-\Psi\left(\frac{x_{n}-1}{2^{n}}\right)+\Psi\left(\frac{x_{n}-1}{2^{n}}\right)-\Psi\left(\frac{x_{n}-2}{2^{n}}\right)+\Psi\left(\frac{x_{n}-2}{2^{n}}\right) \pm \cdots \\
& \cdots-\Psi\left(\frac{1}{2^{n}}\right)+\Psi\left(\frac{1}{2^{n}}\right)-\Psi(0)=\sum_{k=1}^{x_{n}}\left(\Psi\left(\frac{k}{2^{n}}\right)-\Psi\left(\frac{k-1}{2^{n}}\right)\right)
\end{aligned}
$$

Since $\Psi$ is a generalized primitive, for each $k$ it is fulfilled

$$
\begin{equation*}
f\left(\frac{k-1}{2^{n}}\right) \frac{1}{2^{n}} \leq \Psi\left(\frac{k}{2^{n}}\right)-\Psi\left(\frac{k-1}{2^{n}}\right) \leq f\left(\frac{k}{2^{n}}\right) \frac{1}{2^{n}} . \tag{4}
\end{equation*}
$$

Summing the inequalities (4) we obtain

$$
\mu_{n}(f, x)-\frac{1}{2^{n}} f\left(\frac{x_{n}}{2^{n}}\right) \leq \Delta_{n} \leq \mu_{n}(f, x)
$$

Passing to the limit $n \rightarrow \infty$ yields

$$
F(x) \leq \Psi(x)-\Psi(0) \leq F(x)
$$

and

$$
\Psi(x)=F(x)+\Psi(0), x \in[0,1] .
$$

In establishing the last equality we have used the continuity of $\Psi$ as well as Lemma 2.2.
3. Final remarks. Newton integrals as above are easily defined for functions of bounded variation (which are differences of monotone functions). The latter include Lipschitz functions, $C^{1}$ functions, etc. Here it is instructive to recall that a Lipschitz function may not be differentiable only on a countable set of points.

## REFERENCES

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## ПРИМИТИВНИ НА НЯКОИ КЛАСИ ФУНКЦИИ

## Владимир Т. Тодоров, Петър В. Стоев, Михаил М. Константинов

В работата се разглеждат някои проблеми, свързани с преподаването на темата "определен интеграл в смисъла на Нютон" в техническите университети. За да се направи това е необходимо да се разполага с достатъчен запас от функции, интегруеми по Нютон. Показано е, че всяка по части монотонна функция има обобщена примитивна. Изложението на материала изисква минимум предварителни сведения и не се позовава на недоказани твърдения.

