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ON THE INTERVAL ARITHMETIC IN MIDPOINT-RADIUS FORM

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This work presents a collection of basic formulae related to the real interval arithmetic systems $(I(\mathbb{R})^n, +, \mathbb{R}, *, \subseteq)$ and $(I(\mathbb{R}), +, \times, \subseteq)$ under the usage of midpoint-radius presentation.

1. Introduction. It has been recently shown, that interval spaces, when embedded into groups, are direct sums of known number systems for the midpoints of the intervals (like linear spaces, fields, etc) and special spaces abstracting properties of the radii of the intervals [5],[6]. This fact emphasizes the important roles of the midpoint-radius presentation of intervals, both from a theoretical and a practical point of view. In this work we collect a number of formulae for interval arithmetic in midpoint-radius form restricting ourselves to real interval arithmetic.

Denote by $\mathbb{R}$ the set (or field) of reals, by $\mathbb{R}^n$ denote the vector lattice $(\mathbb{R}^n, +, \mathbb{R}, \cdot, \leq)$ with the familiar order, that is for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$, $i = 1, \ldots, n$. Given $u' = ({u'}_1, \ldots, {u'}_n), u'' = (u''_1, \ldots, u''_n) \in \mathbb{R}^n$, $u'' \geq 0$, the set $u = \{\xi \mid u' - u'' \leq \xi \leq u' + u''\}$ is a (compact) n-dimensional *interval* on $\mathbb{R}^n$ with *midpoint (center)* u' and *radius* u''. The *end-point* notation of an interval is: $u = [u' - u'', u' + u'']$. In this work we shall use the *midpoint-radius (center-radius)* presentation of an interval $u \in I(\mathbb{R})^n$ in the form $u = (u'; u'')$, where u', $u'' \in \mathbb{R}^n$, $u'' \geq 0$. The midpoint-radius presentation of intervals will be briefly called *MR-form.* Another common interpretation says that u is an *approximate number (vector)* with a representative (possible, probable) value u' and *error bound* u''. The set of all intervals on $\mathbb{R}^n$ is denoted by $I(\mathbb{R})^n$; in particular, for $n = 1$ the set of intervals on $\mathbb{R}$ is denoted by $I(\mathbb{R})$. An n-dimensional interval is an n-tuple of one-dimensional intervals, symbolically $u = (u_1, u_2, \ldots, u_n)$, $u_i = ({u'}_i; u''_i) \in I(\mathbb{R}), i = 1, 2, \ldots, n$. In the sequel α, β, γ, etc. denote vectors from $\mathbb{R}^n$ (or scalars from $\mathbb{R}$), whereas $a, b, c, \ldots$ denote intervals from $I(\mathbb{R})^n$, and, in particular, from $I(\mathbb{R})$.

2. The interval arithmetic system $(I(\mathbb{R})^n, +, \mathbb{R}, *, \subseteq)$.

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2.1. Addition and multiplication by scalar. Addition and multiplication by scalar are defined for $a, b \in I(\mathbb{R})^n$, $\gamma \in \mathbb{R}$, by $a + b = \{\alpha + \beta \mid \alpha \in a, \beta \in b\}$, $\gamma * b = \{\alpha\beta \mid \beta \in b\}$. In MR-form we have:

$$a + b = (a'; a'') + (b'; b'') = (a' + b'; a'' + b''), \tag{1}$$
$$\gamma * b = \gamma * (b'; b'') = (\gamma b'; |\gamma| b''). \tag{2}$$

Multiplication by -1 (negation): $-1 * b = -1 * (b'; b'') = (-b'; b'')$ is briefly denoted $\neg b = -1 * b$. Subtraction

$$a + (\neg b) = a + (-1) * b = (a' - b'; a'' + b'') \tag{3}$$

is briefly denoted as $a \neg b = a + (-1) * b$. We have $a \neg b = \{\alpha - \beta \mid \alpha \in a, \beta \in b\}$.

Note that the expressions $(a'+b'; a''+b'')$, $(\gamma b'; |\gamma| b'')$, $(a'-b'; a''+b'')$ in the right-hand sides of (1)–(3) are n-dimensional intervals whose midpoints and radii are n-dimensional real vectors to be computed entrywise, e. g. $a' + b' = (a'_1 + b'_1, \ldots, a'_n + b'_n)$, etc. Note also that it does not matter if one presents (1)–(3) as shown above, or presents them first entrywise in $I(\mathbb{R})^n$, i. e. $a + b = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$, $\alpha * b = \alpha * (b_1, b_2, \ldots, b_n) = (\alpha * b_1, \alpha * b_2, \ldots, \alpha * b_n)$, $a \neg b = (a_1 \neg b_1, a_2 \neg b_1, \ldots, a_n \neg b_n)$, and then uses formulae (1)–(3) for the one-dimensional components, namely $a_i + b_i = (a_i{}' + b_i{}'; a''_i + b''_i)$, $\alpha * b_i = (\alpha * b_i{}'; |\alpha| * b''_i)$, $a_i \neg b_i = (a_i{}' - b_i{}'; a''_i + b''_i)$, $i = 1, \ldots, n$.

Note also the presentation: $a = (a'; 0) + (0; a'')$, showing that every interval is a sum of a *point interval* of the form $(u; 0)$ and a *centred (zero-symmetric) interval* of the form $(0; v), v \geq 0$.

2.2. Order and lattice operations in $I(\mathbb{R})^n$. The order relation "inclusion" in center-radius coordinates is:

$$a \subseteq b \Longleftrightarrow |b' - a'| \leq b'' - a'', \quad a, b \in I(\mathbb{R})^n, \tag{4}$$

where, as usually for $\alpha \in \mathbb{R}^n$ we have $|\alpha| = (|\alpha_1|, \ldots, |\alpha_n|)$.

The related lattice operations $x \vee_{\subseteq} y = \sup_{\subseteq}(x, y)$ and $x \wedge_{\subseteq} y = \inf_{\subseteq}(x, y)$ in $I(\mathbb{R})^n$ are obvious in the case when the arguments are ordered, e. g. if $x \subseteq y$, then $x \vee_{\subseteq} y = y$. In the case when neither $x \subseteq y$, nor $y \subseteq x$ hold true, we have:

$$\begin{aligned} x \vee_{\subseteq} y &= (1/2) * (x' + y' + |x'' - y''|;\ |x' - y'| + x'' + y''); \\ x \wedge_{\subseteq} y &= (1/2) * (x' + y' - |x'' - y''|;\ -|x' - y'| + x'' + y''). \end{aligned}$$

In the last formula we assume that $-|x' - y'| + x'' + y'' \geq 0$, that is, the radius of $x \wedge_{\subseteq} y$ is nonnegative; otherwise we have $x \wedge_{\subseteq} y = \emptyset$.

3. The interval arithmetic system $(I(\mathbb{R}), +, \times, \subseteq)$. Addition and order for one-dimensional intervals in $I(\mathbb{R})$ are special cases ($n = 1$) of (1), resp. (4), as defined in $I(\mathbb{R})^n$. We consider next multiplication in $I(\mathbb{R})$ defined by $a \times b = \{\alpha\beta \mid \alpha \in a, \beta \in b\}$, $a, b \in I(\mathbb{R})$. We are looking for a presentation of $a \times b$ in MR-form. We first consider the case when the intervals a, b do not contain zero in their interior.

The set of intervals that do not contain zero as an inner point is denoted $I(\mathbb{R})^*$, the set of intervals that contain zero is denoted $\mathcal{Z}$.

3.1. Multiplication of intervals not containing zero as interior points. Multiplication of a, b that may have zero only as an end-point can be presented in MR-form as follows:

$$a \times b = (a'b' + \tilde{a}''\tilde{b}''; |a'|b'' + |b'|a''), \ a, b \in I(\mathbb{R})^*, \tag{5}$$

where $\tilde{a}'' = \sigma(a')\, a''$, resp. $\tilde{b}'' = \sigma(b')\, b''$ and σ is the "sign" functional: $\sigma(\alpha) = \{-1, \alpha < 0; 1, \alpha \geq 0\}$ for $\alpha \in \mathbb{R}$.

Reciprocal, defined for $b \in I(\mathbb{R}) \setminus \mathcal{Z}$ as $1/b = \{1/\beta \mid \beta \in b\}$, is given in MR-form by:

$$1/b = (\delta^2 b'; \delta^2 b'') = \delta^2 * b, \ \ \delta^2 = \delta^2(b) = ({b'}^2 - {b''}^2)^{-1}, \ b \in I(\mathbb{R}) \setminus \mathcal{Z}. \tag{6}$$

Division $a \times (1/b) = \{\alpha/\beta \mid \alpha \in a, \beta \in b\}$ for $a \in I(\mathbb{R})^*, \ b \in I(\mathbb{R}) \setminus \mathcal{Z}$ is presented in MR-form by:

$$\begin{aligned} a \times (1/b) &= (\delta^2(a'b' + \tilde{a}''\tilde{b}''); \delta^2(|a'|b'' + |b'|a'')) \\ &= \delta^2 * (a'b' + \tilde{a}''\tilde{b}''; |a'|b'' + |b'|a''), \\ &= \delta^2 * (a \times b). \end{aligned} \tag{7}$$

To consider multiplication and division in the cases, when some of the arguments a, b contain zero in their interior, we need some special functionals.

3.2. The functionals $\boldsymbol{\kappa}$ and $\boldsymbol{\chi}$. The functionals κ and χ are defined for $a = (a'; a'') \in I(\mathbb{R})$ by:

$$\kappa(a) = a''/|a'|, \ \ a' \neq 0; \tag{8}$$

$$\chi(a) = (|a'| - a'')/(|a'| + a''), \ \ a \neq 0. \tag{9}$$

Using κ the condition $a \in I(\mathbb{R})^*$ can be equivalently written as $\kappa(a) \leq 1$, and $a \in I(\mathbb{R}) \setminus \mathcal{Z}$ can be written as $\kappa(a) < 1$.

The function χ is defined for all intervals $a \neq 0$ and for the range of χ we have $-1 \leq \chi \leq 1$; the function κ is defined for $a \in I(\mathbb{R})$, such that $a' \neq 0$ and we have $0 \leq \kappa < \infty$. The functions χ and κ are related as follows (in their common definition domain):

$$\chi(a) = \frac{1 - \kappa(a)}{1 + \kappa(a)}, \quad \kappa(a) = \frac{1 - \chi(a)}{1 + \chi(a)}.$$

Both functions χ and κ are constant on every *radial line* $t * a = (ta'; |t|a''), \ \ t \in [0, \infty]$, starting at the origin and passing through a fixed interval $a \in I(\mathbb{R}), a \neq 0$. In fact χ and κ have the same value also on the radial line starting at the origin and passing through the interval $\neg a$. The union of the two radial lines, called *interval axis* [3], is given by $t * a = (ta'; |t|a''), \ t \in \mathbb{R}$. Thus, the values $\chi(t * a)$ and $\kappa(t * a)$ on every interval axis do not depend on $t \in \mathbb{R}$. Therefore it is sufficient to consider χ and κ only on the unit semi-circle of the form $a_\varphi = (\cos\varphi; \sin\varphi)$, $\varphi \in [0, \pi]$. Then from (8) we have

$$\kappa(a_\varphi) = \kappa(\varphi) = \kappa(\cos\varphi; \sin\varphi) = |\tan\varphi|, \ \ \varphi \in [0, \pi].$$

From (9) we obtain

$$\chi(\varphi) = \frac{|\cos\varphi| - \sin\varphi}{|\cos\varphi| + \sin\varphi} = \frac{1 - |\tan\varphi|}{1 + |\tan\varphi|}, \quad \varphi \in [0, \pi].$$

The functions χ and κ are symmetric in the interval $[0, \pi]$ with respect to the midpoint $\pi/2$, that is, for $\varphi \in [0, \pi/2]$ we have: $\kappa(\varphi) = \kappa(\pi - \varphi)$, resp. $\chi(\varphi) = \chi(\pi - \varphi)$.

The function $\kappa(\varphi) = \kappa(\cos\varphi; \sin\varphi)$ is an increasing function on φ in $[0, \pi/2]$ and χ is decreasing in $[0, \pi/2]$. We have: $\chi(a) \leq \chi(b) \Longleftrightarrow \kappa(a) \geq \kappa(b)$.

This concludes the characterization of the functionals $\kappa : [0, \pi] \longrightarrow [0, \infty)$ and $\chi : [0, \pi] \longrightarrow [-1, 1]$, as needed for our further purposes.

Remark. It is sufficient to consider one of the functions χ and κ, but as both functions are used in practice we introduced them simultaneously. The function χ is introduced and studied by H. Ratschek, see e. g. [9], whereas κ in slightly different definitions is introduced and used by Z. Kulpa, see e. g. [3] and S. Rump [10].

3.3. Multiplication of intervals containing zero in their interior. If $a = (a'; a'')$ is such that $a'' < |a'|$ (that is, $\kappa(a) > 1$), then, under the condition that either: i) $b'' \leq |b'|$ (that is $\kappa(b) \leq 1$), or ii) $b'' > |b'|$, $\kappa(a) \geq \kappa(b)$ (that is $\kappa(a) \geq \kappa(b) > 1$, equivalently $\chi(a) \leq \chi(b) < 0$), we have:

$$\begin{aligned} (a'; a'') \times (b'; b'') &= (a'b' + \sigma(b')a'b'';\ |b'|a'' + a''b'') \\ &= (a'(b' + \sigma(b')b'');\ a''(|b'| + b'')) \\ &= (\sigma(b')a'(|b'| + b'');\ a''(|b'| + b'')). \end{aligned} \tag{10}$$

Remark. The condition $\kappa(a) \geq \kappa(b) > 1$, equivalently $\chi(a) \leq \chi(b) < 0$, in ii) above presents no restriction, as, if not satisfied, one has to interchange a and b.

Using multiplication by scalar (2) formula (10) can be written, using the notation $|b| = |b'| + b''$, as:

$$(a'; a'') \times (b'; b'') = (b' + \tilde{b}'') * (a';\ a'') = \sigma(b')|b| * (a'; a''). \tag{11}$$

We can now summarize the various cases (5), (11) in a general formula:

$$a \times b = \begin{cases} (a'b' + \tilde{a}''\tilde{b}''; |a'|b'' + |b'|a''), & \kappa(a) \leq 1,\ \kappa(b) \leq 1; \\ \sigma(b')|b| * (a'; a''), & \kappa(a) > 1 \geq \kappa(b) \text{ or } \kappa(a) \geq \kappa(b) > 1; \\ \sigma(a')|a| * (b'; b''), & \kappa(b) > 1 \geq \kappa(a) \text{ or } \kappa(b) \geq \kappa(a) > 1. \end{cases} \tag{12}$$

In the case when $0 \in a$, that is $\kappa(a) \geq 1$, and $\kappa(b) < 1$, the formula for division (7) obtains the form [1]:

$$a \times (1/b) = \sigma(b')|1/b| * (a';\ a'') = \sigma(b')\delta^2(b)|b| * (a';\ a''). \tag{13}$$

Summarizing (7) and (13), we see that division (whenever defined) is reduced to multiplication:

$$a \times (1/b) = \delta^2(b) * (a \times b), \quad \kappa(b) < 1.$$

4. Centred (outer) interval multiplication and division. The *centred (outer) multiplication* (*co-multiplication*) is defined as:

$$u \circ v = (u'v'; |v'|u'' + |u'|v'' + u''v''). \tag{14}$$

We have $u \times v \subseteq u \circ v$, cf. [3], [10]. Note that the midpoint of (14) is the product of the *midpoints* of the arguments. For comparison the midpoint of (12) is more complex.

The *centred (outer) division* (*co-division*), is defined as $u \circ (1/v)$, providing $\kappa(v) < 1$. Using (6) we have:

$$u \circ (1/v) = u \circ (\delta^2 * v) = \delta^2 * (u'v'; u''|v'| + |u'|v'' + u''v'') = \delta^2 * (u \circ v), \tag{15}$$

where $u \in I(\mathbb{R}),\ v \in I(\mathbb{R}) \backslash \mathcal{Z}$. This shows that co-division is reduced to co-multiplication.

More relations involving κ and χ. For all $a, b \in I(\mathbb{R})^*$ the following relations hold true:

$$\kappa(a \times b) = \frac{\kappa(a) + \kappa(b)}{1 + \kappa(a)\kappa(b)}, \quad \chi(a \times b) = \chi(a)\chi(b).$$

We have for $a, b \in I(\mathbb{R})$:

$$\begin{aligned} \kappa(a \circ b) &= \kappa(a) + \kappa(b) + \kappa(a)\kappa(b), \\ \chi(a \circ b) &= (1/2)(\chi(a) + \chi(b) + \chi(a)\chi(b) - 1). \end{aligned}$$

5. Notes and conclusions. The centred (outer) interval multiplication (14) has been proposed in [12]; independently it has been introduced and practically implemented in [11], cf. also [7]. A. Neumaier [8] studied (14) with respect to distributivity. As noted in [8], the operation (14) is a special case of the complex disc multiplication introduced in [2]. Further study and software implementation of the co-multiplication is reported in [10], where the relation $(u \circ v)''/(u \times v)'' \leq 1.5$ is proved to hold. Inclusion monotonicity and other properties of co-multiplication are studied in [3] and [4].

The operation (14) has a simple definition and leads to simple solutions of interval algebraic problems like interval linear systems [4]. At the same time (14) approximates the set-theoretic multiplication (12) rather well, especially for narrow intervals which are the most common in practical applications. As shown in [10] computer implementations of (14) present no problem. The operation (14) also plays important roles in the abstract study of interval spaces [5].

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ВЪРХУ ИНТЕРВАЛНАТА АРИТМЕТИКА В ЦЕНТЪР-РАДИУС ФОРМА

Светослав Марков, Далсидио Клаудио

Настоящата работа е колекция от основни формули отнасящи се до реалните интервално-аритметични системи $(I(\mathbb{R})^n, +, \mathbb{R}, *, \subseteq)$ и $(I(\mathbb{R}), +, \times, \subseteq)$ при използване на център-радиус представяне.