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EXTENSION OF HOLOMORPHIC FUNCTIONS

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In several complex variables it is well known that a separately holomorphic function on a product of domains is already continuous and therefore holomororphic. We will discuss separately holomorphic function on a more complicated set and study the associated maximal domain into which all such functions are holomorphically extendible.

In the theory of one complex variable it is well known that for any domain $G \subset \mathbb{C}$ there is an $f \in \mathcal{O}(G)$ (i.e. f is a holomorphic function on G) that cannot be holomorphically extended beyond G. In many variables the situation becomes different. There are pairs of domains $D_1 \underset{\neq}{\subsetneq} D_2 \subset \mathbb{C}^n$, $n \geq 2$, such that the restriction mapping $\mathcal{O}(D_2) \longrightarrow \mathcal{O}(D_1)$ is surjective. Domains which carry a non extendible holomorphic function are called *domains of holomorphy*. They can be characterized by being *pseudoconvex*, i.e. $-\log \operatorname{dist}(\cdot, \partial D)$ is a plurisubharmonic function.

Moreover, in contrast to the case of real partial differentiability, a separately holomorphic (i.e. a partially complex differentiable) function $f : D \longrightarrow \mathbb{C}$ (write $f \in \mathcal{O}_s(D)$) — $D \subset \mathbb{C}^n$, $n \geq 2$, a domain — is already continuous (Theorem of Hartogs (1906)) and, therefore, using the Cauchy integral formula holomorphic on D. Recall that a function $f : D \longrightarrow \mathbb{C}$ is called *separately holomorphic* if for any $a \in D$ and any $j \in \{1, \dots, n\}$ the function of one complex variable $f(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_n) : D_{a,j} \longrightarrow \mathbb{C}$ is holomorphic, where

$$D_{a,j} := \{\lambda \in \mathbb{C} : (a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n) \in D\}.$$

In this lecture we will present an extension theorem which may be thought as a generalization of the above quoted theorem of Hartogs. More details may be found in [1].

Let $N \in \mathbb{N}$ and let $A_j \subset D_j \subset \mathbb{C}^{k_j}$, D_j a domain, $j = 1, \dots, N$. The following set

$$X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) := \bigcup_{j=1}^{N} \Big(A_1 \times \dots \times A_{j-1} \times D_j \times A_{j+1} \times \dots \times A_N \Big)$$

is called *the N-fold cross associated to the N pairs $(A_j, D_j)_j$*.

Moreover, let $M \subset X$ ($M = \varnothing$ is allowed). For a point $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and a j, $1 \leq j \leq N$, we define *the fiber of M over* $(a_1, \dots, \widehat{a_j}, \dots, a_N)$ as

$$M_{(a_1, \dots, \widehat{a_j}, \dots, a_N)} := \{z_j \in D_j : (a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_N) \in M\},$$

where $(a_1, \dots, \widehat{a_j}, \dots a_N) := (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_N)$.

We will always assume that all the fibers $M_{(a_1, \dots, \widehat{a_j}, \dots, a_N)}$ are closed in D_j.

Then a function
$$f : \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) \setminus M \longrightarrow \mathbb{C}$$
is called *separately holomorphic on* $X \setminus M$ if
$$\forall_{(a_1,\dots,a_N)\in A_1\times\dots\times A_N}, \forall_{1\le j\le N}$$
$$f(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_N) \in \mathcal{O}(D_j \setminus M_{(a_1,\dots,\widehat{a_j},\dots,a_N)}).$$
We write $f \in \mathcal{O}_s(X \setminus M)$.

Problem. *Let N, A_j, D_j, and M be as above. What are conditions on these sets under which there exist a (pseudoconvex) domain $\widehat{X} \subset \mathbb{C}^{k_1+\dots+k_N}$, $X \subset \widehat{X}$, a large subset $X' \subset X$, and a relatively closed subset $\widehat{M} \subset \widehat{X}$, $\widehat{M} \cap X' \subset M$, such that:*
$$\forall_{f\in\mathcal{O}_s(X\setminus M)} \exists_{\widehat{f}\in\mathcal{O}(\hat{X}\setminus\widehat{M})} : \widehat{f}|_{X'\setminus M} = f|_{X'\setminus M},\ \widehat{f} \text{ uniquely determined.}$$

Before stating the theorem let us recall a few definitions:

Let $A \subset G \subset \mathbb{C}^n$, where G is a domain. Define $h^*_{A,G}$ as the upper continuous regularization of $h_{A,G} := \sup\{u \in \mathcal{PSH}(G) : u \le 1, u|_A \le 0\}$ – $\mathcal{PSH}(G)$ denotes the set of all plurisubharmonic functions on G. $h^*_{A,G}$ is the so called *relative extremal function of the pair* (A, G).

A set $A \subset G$, G a domain in $\mathbb{C}^n$, is called *locally pluriregular*, if for any $a \in A$ and any neighborhood $U = U(a)$ we have: $h^*_{A\cap U,U}(a) = 0$. Observe that such a set is "thick" in the pluripotential sense; in particular, it is not pluripolar.

Moreover, if $G_j \nearrow G$, then $\omega_{A,G} := \lim_{j\to\infty} h^*_{A\cap G_j,G_j}$. Note that the definition of $\omega_{A,G}$ is independent of the exhaustion sequence and that, if G is bounded, then $\omega_{A,G} = h^*_{A,G}$.

Let $X = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, , D_N)$ and $M \subset X$. Put
$$\widehat{X} := \{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \omega_{A_1,D_1}(z_1) + \dots + \omega_{A_N,D_N}(z_N) < 1\}.$$
Observe that $X \subset \widehat{X}$ and if all the D_j's are pseudoconvex, then $\widehat{X}$ is a pseudoconvex domain.

Moreover, set
$$\Sigma_j := \{(a', a'') \in (A_1 \times \dots \times A_{j-1}) \times (A_{j+1} \times \dots \times A_N) : M_{(a_1,\dots,\widehat{a_j},\dots,a_N)} \text{ not pluripolar}\}.$$
In the sequel we will be interested in the case when the Σ_j's are pluripolar. A pluripolar set may be thought as a very thin set.

We introduce the following modified N-fold cross
$$X' := \bigcup_{j=1}^{N} \{(z', z_j, z'') \in A_1 \times \dots \times A_{j-1} \times D_j \times A_{j+1} \times \dots \times A_N : (z', z'') \notin \Sigma_j\}.$$

With these notions we have the following result (cf. [2]):

Theorem. *Let N, A_j, D_j be as before. Assume that the D_j's are pseudoconvex and the A_j's locally pluriregular. Put $X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$. Let $M \subset X$ be a relatively closed subset of X such that Σ_j is pluripolar, $j = 1, \dots, N$. Then there exists a pluripolar set $\widehat{M} \subset \widehat{X}$, relatively closed, satisfying the following properties:*

- *$\widehat{M} \cap X' \subset M$,*

• *for any function* $f \in \mathcal{O}_s(X \setminus M)$ *there exists a unique extension* $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ *such that* $\widehat{f} = f$ *on* $X' \setminus M$,

• $\widehat{X} \setminus \widehat{M}$ *is pseudoconvex.*

Remark. (a) If M is pluripolar, then the assumptions are fulfilled.

(b) In the case that all fibers $M_{(a_1,\dots,\widehat{a_j},\dots,a_N)}$ are pluripolar we obviously have $X' = X$.

(c) There is an analogous result in the case when the singularity set M is an analytic set in some neighborhood of X (see [3]); in this case the set $\widehat{M}$ is again an analytic set in $\widehat{X}$ and X' could be taken equal to X.

Example (a) Let $D_j := E \subset \mathbb{C}$ be the open unit disc in the complex plane, $A_j := D_j$, $j = 1, \dots, N$, and $M = \varnothing$. Then $\mathcal{O}_s(X) = \mathcal{O}(\widehat{X})$, where

$$X = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) = \widehat{X} \subset \mathbb{C}^n,$$

i.e. the theorem of Hartogs is a particular case of the above theorem.

(b) Let $N = 2$, $D_1 = D_2 := \mathbb{B}(0, R) \subset \mathbb{C}^n$, where $\mathbb{B}(0, r) := \{z \in \mathbb{C}^n : \|z\| < R\}$ denotes the Euclidean ball with center at 0 and radius R, and $M = \varnothing$. For an $r \in (0, R)$ put $A_1 = A_2 = \partial\mathbb{B}(0, r)$ and set

$$X := \mathbb{X}(A_1, A_2; D_1, D_2) = \Big(\partial\mathbb{B}(0, r) \times \mathbb{B}(0, R)\Big) \cup \Big(\mathbb{B}(0, R) \times \partial\mathbb{B}(0, r)\Big).$$

Then the above theorem gives that any $f \in \mathcal{O}_s(X)$ is the restriction of an $\widehat{f} \in \mathcal{O}(\widehat{X})$, where

$$\widehat{X} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n : \max\{\|z\|, \|w\|\} < R, \|z\|\|w\| < rR\}.$$

Observe that we have always holomorphic continuation from X to the much larger domain $\widehat{X}$, i.e. the theorem may be understood also in the sense of simultaneous holomorphic extension as mentioned at the very beginning.

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ПРОДЪЛЖЕНИЕ НА ХОЛОМОРФНИ ФУНКЦИИ

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В теорията на функциите на много комплексни променливи е добре известно, че всяка разделно холоморфна функция, дефинирана върху произведение на области, е непрекъсната и следователно холоморфна. В доклада се разглеждат разделно холоморфни функции, дефинирани върху по-сложно множество, като се изучава асоциираната максимална област, върху която всички такива функции се продължават холоморфно.