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ONE-DIMENSIONAL SHAPE SPACES*

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Using equivalence classes of similar triangles H. Sato introduces a closed convex curve associated to a non-degenerate triangle. We show that this curve is a circle in the Lester's model of the two-dimensional shape space. We also prove that all such circles form a Poncelet pencil.

There are different ways to investigate the equivalence classes of triangles with respect to the group $G = \mathrm{Sim}^+(\mathbb{R}^2)$ of the direct similarities of the Euclidean plane $\mathbb{R}^2$. One of these ways is due to H. Sato (see [5]). For a fixed non-degenerate triangle $\triangle abc$, he considered a point (x, y, z) in the Euclidean space $\mathbb{R}^3$, where $x = \sphericalangle(bac)$, $y = \sphericalangle(cba)$, $z = \sphericalangle(acb)$. Thus the points of the set

$$\Pi = \{(x, y, z) \mid x + y + z = \pi,\ x > 0, y > 0, z > 0\}$$

represent the equivalence classes of similar triangles in $\mathbb{R}^2$. Let $a(t)$, $b(t)$, $c(t)$ be points lying on the sides $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ of $\triangle abc$ such that the corresponding affine ratios are $(aba(t)) = (bcb(t)) = (cac(t)) = t : (1 - t)$. H. Sato proves that the set of non-degenerate triangles $\triangle abc$

$$T(\triangle\, abc) = \{\triangle\, a(t)b(t)c(t) \mid t \in \mathbb{R}\}$$

is represented by a closed convex curve in Π.

Another representation of the classes of similar triangles is the Euclidean plane extended with a point at infinity. This interpretation is realized by J. Lester in [4]. For that purpose, the Euclidean plane is identified with the field of complex numbers $\mathbb{C}$ and it is extended by a point at infinity, i. e. $\mathbb{C}_\infty = \mathbb{C} \bigcup \infty$. Let us recall some basic facts from [4]. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three points in $\mathbb{C}$ and at most two of them are coinciding, then it is defined a triangle $\triangle\mathbf{abc}$ .Degenerated triangles with distinct collinear vertices or two coinciding vertices are allowed. There exists a complex number which determines the ordered triangle $\triangle\mathbf{abc}$ up to a direct plane similarity. According to [4], this is the number

$$\triangle_{\mathbf{abc}} = \frac{\mathbf{a} - \mathbf{c}}{\mathbf{a} - \mathbf{b}} \in \mathbb{C}_\infty, \tag{1}$$

called a shape of the triangle $\triangle\mathbf{abc}$. In particular, $\triangle\mathbf{abc}$ is isosceles with apex at $\mathbf{a}$ whenever $|\triangle_{\mathbf{abc}}| = 1$, $\triangle\mathbf{abc}$ is equilateral when $\triangle_{\mathbf{abc}} = \omega = \frac{1}{2} + i.\frac{\sqrt{3}}{2}$ or $\triangle_{\mathbf{abc}}$

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$= \overline{\omega} = \dfrac{1}{2} - i.\dfrac{\sqrt{3}}{2}$ and $\triangle\mathbf{abc}$ is right-angled at $\mathbf{a}$ whenever $\triangle_{\mathbf{abc}}$ is imaginary. It is clear that $\triangle_{\mathbf{abc}} = \infty \iff \mathbf{a} = \mathbf{b} \neq \mathbf{c}$. For any degenerate triangle with $\mathbf{a} \neq \mathbf{b}$, $\triangle_{\mathbf{abc}} \in \mathbb{R}$.

D. Kendall introduced the notion of the two-dimensional shape space in [2]. The set Π and the extended plane $\mathbb{C}_\infty$ are models of this shape space. We call them the Sato's model and the Lester's model, respectively. In this paper we obtain a representation of the set $T(\triangle abc)$ in the Lester's model. Then this representation can be considered as a one-parameter set of triangle shapes or as a one-dimensional shape space. Moreover we shall describe all such one-dimensional shape spaces.

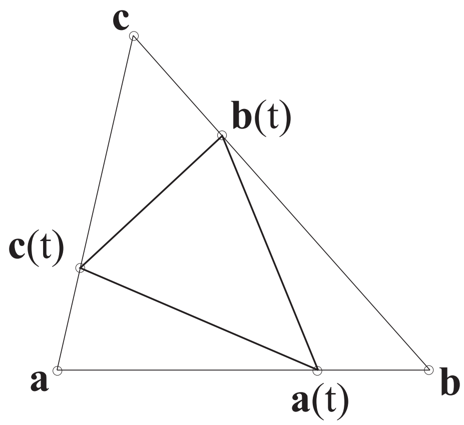

Fig. 1

Let $\mathbf{z} \in \mathbb{C}$ be the shape of the triangle $\triangle\mathbf{abc}$, i. e. $\triangle_{\mathbf{abc}} = \mathbf{z}$. Without loss of generality we may suppose $\mathbf{a} = 0$, $\mathbf{b} = 1$, $\mathbf{c} = \mathbf{z}$. If the points $\mathbf{a}(t) \in \overline{\mathbf{ab}}$, $\mathbf{b}(t) \in \overline{\mathbf{bc}}$ and $\mathbf{c}(t) \in \overline{\mathbf{ca}}$ (see Fig. 1) are such that $\mathbf{a}(t) = (1-t)\mathbf{a} + t\mathbf{b}$, $\mathbf{b}(t) = (1-t)\mathbf{b} + t\mathbf{c}$, $\mathbf{c}(t) = (1-t)\mathbf{c} + t\mathbf{a}$, where $t \in \mathbb{R}$, then $\mathbf{a}(t) - \mathbf{c}(t) = (1-t)(\mathbf{a}-\mathbf{c}) + t(\mathbf{b}-\mathbf{a})$ $= [(1-t)\mathbf{z} - t](\mathbf{a}-\mathbf{b})$ and $\mathbf{a}(t) - \mathbf{b}(t) = (1-t)(\mathbf{a}-\mathbf{b}) + t(\mathbf{b}-\mathbf{c}) = (1-2t)(\mathbf{a}-\mathbf{b})$ $+t(\mathbf{a}-\mathbf{c}) = (1-2t+t\mathbf{z})(\mathbf{a}-\mathbf{b})$. Using (1), we find that

$$\mathbf{w} = \triangle_{\mathbf{a}(t)\mathbf{b}(t)\mathbf{c}(t)} = \frac{(1-t)\mathbf{z} - t}{t\mathbf{z} + 1 - 2t}, \quad t \in \mathbb{R}. \tag{2}$$

Three distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{C}$ define commonly six distinct ordered triangles. The triangles $\triangle\mathbf{abc}$, $\triangle\mathbf{bca}$ and $\triangle\mathbf{cab}$ have the same orientation and different shapes. We obtain the shapes of the triangles $\triangle_{\mathbf{abc}} = \mathbf{z}$, $\triangle_{\mathbf{bca}} = \dfrac{1}{1-\mathbf{z}}$ and $\triangle_{\mathbf{cab}} = 1 - \dfrac{1}{\mathbf{z}}$ replacing t in (2) by 0, 1 and 1/2, respectively.

The equation (2), obtained above, allows us to define a map of $\mathbb{R}$ into the extended Euclidean plane. Let $\mathbf{z} \in \mathbb{C}\backslash\{\mathbb{R} \cup \omega \cup \overline{\omega}\}$ be fixed. We consider the map $\varphi_z : \mathbb{R} \longrightarrow \mathbb{C} \setminus \mathbb{R}$ such that

$$\varphi_z(t) = \frac{(1-t)\mathbf{z} - t}{t\mathbf{z} + 1 - 2t}, \quad t \in \mathbb{R}. \tag{3}$$

Then, φ_z is a curve in the Euclidean plane which corresponds to a one-parameter family of triangle shapes. In other words, φ_z represents a one-dimensional shape space. Moreover, $\mathbf{z} \in \varphi_z$, $\dfrac{1}{1-\mathbf{z}} \in \varphi_z$ and $\dfrac{\mathbf{z}-1}{\mathbf{z}} \in \varphi_z$.

Proposition 1. *The curve φ_z, defined by* (3) *is a circle in $\mathbb{C}\setminus\mathbb{R}$, passing through the points* $\mathbf{z}$, $\dfrac{1}{1-\mathbf{z}}$ *and* $\dfrac{\mathbf{z}-1}{\mathbf{z}}$.

Proof. It is well known that four points $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ in $\mathbb{C}$ are concyclic or collinear if and only if the cross ratio $[\mathbf{p},\, \mathbf{q};\, \mathbf{r},\, \mathbf{s}] = \dfrac{(\mathbf{p}-\mathbf{r})(\mathbf{q}-\mathbf{s})}{(\mathbf{p}-\mathbf{s})(\mathbf{q}-\mathbf{r})}$ is real. From $\triangle_{\mathbf{z}\frac{1}{1-\mathbf{z}}\frac{\mathbf{z}-1}{\mathbf{z}}} = \dfrac{\mathbf{z}-\frac{\mathbf{z}-1}{\mathbf{z}}}{\mathbf{z}-\frac{1}{1-\mathbf{z}}} = \dfrac{\mathbf{z}-1}{\mathbf{z}} \in \mathbb{C}\setminus\mathbb{R}$ it follows that the points $\mathbf{z}$, $\dfrac{1}{1-\mathbf{z}}$ and $\dfrac{\mathbf{z}-1}{\mathbf{z}}$ are not collinear. Besides,

$$\triangle_{\varphi_z(t)\frac{\mathbf{z}-1}{\mathbf{z}}\frac{1}{1-\mathbf{z}}} = \frac{\varphi_z(t)-\dfrac{\mathbf{z}-1}{\mathbf{z}}}{\varphi_z(t)-\dfrac{1}{1-\mathbf{z}}} = \frac{\dfrac{(1-t)\mathbf{z}-t}{t\mathbf{z}+1-2t}-\dfrac{1}{1-\mathbf{z}}}{\dfrac{(1-t)\mathbf{z}-t}{t\mathbf{z}+1-2t}-\dfrac{\mathbf{z}-1}{\mathbf{z}}} = \frac{1-t}{1-2t}.\frac{\mathbf{z}}{\mathbf{z}-1}.$$

Hence,

$$[\varphi_z(t),\, \mathbf{z};\, \frac{1}{1-\mathbf{z}},\, \frac{\mathbf{z}-1}{\mathbf{z}}] = \triangle_{\mathbf{z}\frac{1}{1-\mathbf{z}}\frac{\mathbf{z}-1}{\mathbf{z}}}.\triangle_{\varphi_z(t)\frac{\mathbf{z}-1}{\mathbf{z}}\frac{1}{1-\mathbf{z}}} = \frac{1-t}{1-2t} \in \mathbb{R}$$

for $t \neq 1/2$. Since $\varphi_z(1/2) = \dfrac{\mathbf{z}-1}{\mathbf{z}} \in \varphi_z$, the proof is completed. $\square$

Proposition 2. $\mathbf{w} \in \varphi_z \Longleftrightarrow \mathbf{z} \in \varphi_{\mathrm{W}}$.

Proof. First we shall prove that $\varphi_z \equiv \varphi_{\mathrm{W}}$ if $\mathbf{w} \in \varphi_z$. From $\mathbf{w} \in \varphi_z$ it follows that there exists $t \in \mathbb{R}$ such that $\mathbf{w} = \dfrac{(1-t)\mathbf{z}-t}{t\mathbf{z}+1-2t}$. Since

$$\frac{1}{1-\mathbf{w}} = \frac{1}{1-\dfrac{(1-t)\mathbf{z}-t}{t\mathbf{z}+1-2t}} = \frac{t\mathbf{z}+1-2t}{(2t-1)\mathbf{z}+1-t} = \frac{(1-t)\frac{1}{1-\mathbf{z}}-t}{t\frac{1}{1-\mathbf{z}}+1-2t},$$

then $\dfrac{1}{1-\mathbf{w}} \in \varphi_{\frac{1}{1-z}} \equiv \varphi_z$. Similarly, $\dfrac{\mathrm{w}-1}{\mathrm{w}} \in \varphi_z$. Since the circle φ_{W} is unique, we obtain that $\varphi_z \equiv \varphi_{\mathrm{W}}$. Hence, if $\mathbf{w} \in \varphi_z$, then $\mathbf{z} \in \varphi_z \equiv \varphi_{\mathrm{W}}$ and vice versa. $\square$

Corollary 1. *Let $\mathbf{z}_i \in \mathbb{C}\setminus\mathbb{R}$, $\mathbf{z}_i \neq \omega, \overline{\omega}$, $i = 1, 2$. Then the circles φ_{z_1} and φ_{z_2}, defined by* (3) *are either coinciding or non-intersecting.*

Proof. The case $\mathbf{z}_1 = \mathbf{z}_2$ is trivial. Let $\mathbf{z}_1 \neq \mathbf{z}_2$. If either $\mathbf{z}_2 \in \{\dfrac{1}{1-\mathbf{z}_1}, \dfrac{\mathbf{z}_1-1}{\mathbf{z}_1}\}$ or $\mathbf{z}_1 \in \{\dfrac{1}{1-\mathbf{z}_2}, \dfrac{\mathbf{z}_2-1}{\mathbf{z}_2}\}$ then $\varphi_{z_1} \equiv \varphi_{z_2}$. Otherwise, let there exists $\mathbf{w} \in \varphi_{z_1} \bigcap \varphi_{z_2}$ and $\varphi_{z_1} \not\equiv \varphi_{z_2}$, i. e. $\mathbf{w} \in \varphi_{z_1}$ and $\mathbf{w} \in \varphi_{z_2}$. Applying Proposition 2 we obtain $\varphi_{z_1} \equiv \varphi_{\mathrm{W}} \equiv \varphi_{z_2}$ which is a contradiction. $\square$

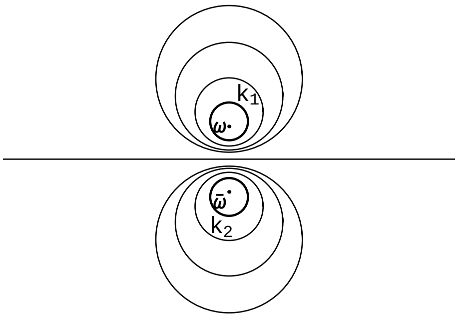

Fig. 2. A Poncelet pencil

When $\mathbf{z} \in \{\omega, \overline{\omega}\}$ the map φ_z is constant, i. e. $\varphi_\omega(t) = \omega$ and $\varphi_{\overline{\omega}}(t) = \overline{\omega}$ for any $t \in \mathbb{R}$. The real line is an axis of symmetry for the set of all circles φ_z, $\mathbf{z} \in \mathbb{C} \setminus \mathbb{R}$. Then, taking into account the above assertion, we prove the main result in this paper.

Theorem 1. *The set Σ of all one-dimensional shape subspaces φ_z, defined by* (3) *for $\mathbf{z} \in \mathbb{C} \setminus \mathbb{R}$, is a Poncelet pencil of circles with limit points ω and $\overline{\omega}$ excepting the radical axis* (see Fig. 2)

The definition and the properties of the Poncelet pencils of circles are known from [1] and [3].

We can find the center $\mathbf{w}_0$ of the circle φ_z, $\mathbf{z} \in \mathbb{C} \setminus \mathbb{R}$, $\mathbf{z} \neq \omega, \overline{\omega}$ using the inversion $\mathbf{z} \longrightarrow \mathbf{w}_0 + \dfrac{R^2}{\overline{\mathbf{z}} - \overline{\mathbf{w}}_0}$, where R is the radius of φ_z. So, the point $\mathbf{w}_0$ is the solution of the equation $[\mathbf{w}_0, \mathbf{z}; \dfrac{1}{1-\mathbf{z}}, \dfrac{\mathbf{z}-1}{\mathbf{z}}] = \overline{[\infty, \mathbf{z}; \dfrac{1}{1-\mathbf{z}}, \dfrac{\mathbf{z}-1}{\mathbf{z}}]} = \overline{\triangle}_{\mathbf{z}\frac{1}{1-\mathbf{z}}\frac{\mathbf{z}-1}{\mathbf{z}}} = \dfrac{\overline{\mathbf{z}}-1}{\overline{\mathbf{z}}}$. Hence $\mathbf{w}_0 = \dfrac{\mathbf{z} - |\mathbf{z}|^2 - 1}{\mathbf{z} - \overline{\mathbf{z}}}$. If $(x, y) \in \mathbb{R}^2$ are the Cartesian coordinates of the point $\mathbf{z} \in \mathbb{R}^2 \cong \mathbb{C}$, i. e. $\mathbf{z} = x + i.y$, then the Cartesian coordinates $(x_{\mathrm{w}_0}, y_{\mathrm{w}_0})$ of the point $\mathbf{w}_0$ are

$$\left(\frac{1}{2}, \frac{1 - x + x^2 + y^2}{2y}\right).$$

For the radius R of φ_z we get

$$R = |\,\mathbf{z} - \mathbf{w}_0| = \left|\frac{\mathbf{z}^2 - \mathbf{z} + 1}{\mathbf{z} - \overline{\mathbf{z}}}\right| = \frac{1}{2|y|}\sqrt{(x^2 - y^2 - x + 1)^2 + y^2(2x-1)^2}.$$

The imaginary line in $\mathbb{R}^2 \cong \mathbb{C}$, representing all right-angled triangles in the plane, has at most two common points with any circle of Σ. Therefore, the Poncelet pencil of circles Σ can be divided into three subset Σ_0, Σ_1 and Σ_2 of one-dimensional shape subspaces, containing 0, 1 or 2 right-angled triangles, respectively. Since $R = \dfrac{1}{2} \Leftrightarrow y^2 = (x^2 - y^2 - x + 1)^2 + y^2(2x-1)^2 \Leftrightarrow (1 - x + x^2 + y^2)^2 = 4y^2 \Leftrightarrow y_{\mathrm{w}_0} = \pm 1$ we have that Σ_1 has only two circles k_1, k_2 with centers $c_1(1/2, 1)$ and $c_2(1/2, -1)$, respectively. These circles are the one-dimensional shape spaces associated to the isosceles right-angled triangles. Thus the centers of the circles of Σ_0 are between the points c_1 and c_2. Otherwise, we get Σ_2.

In [5], H. Sato does not explore the case when the triangles $\triangle\mathbf{abc}$ are degenerated. Having in mind the previous considerations we may examine this case. If the triangle

$\triangle\mathbf{abc}$ is degenerated, i.e. $\triangle_{\mathbf{abc}} = \mathbf{z} \in \mathbb{R}$, then the triangle $\triangle\,\mathbf{a}(t)\mathbf{b}(t)\mathbf{c}(t)$ is also degenerated and

$$\mathbf{w} = \triangle_{\mathbf{a}(t)\mathbf{b}(t)\mathbf{c}(t)} = \frac{(1-t)\mathbf{z}-t}{t(\mathbf{z}-2)+1} \in \mathbb{R} \quad \text{for any } \mathbf{z},\ t \in \mathbb{R}.$$

Since $t = \dfrac{\mathbf{z}-\mathbf{w}}{\mathbf{w}(\mathbf{z}-2)+\mathbf{z}+1}$, the one-dimensional shape space associated to the degenerated $\triangle\,\mathbf{abc}$ is either $\mathbb{R}\cup\infty$ when $\triangle_{\mathbf{abc}} = \mathbf{z} \in \mathbb{R}\setminus\{2\}$ or $\mathbb{R}$ when $\triangle_{\mathbf{abc}} = \mathbf{z} = 2$. Finally, we may conclude that all one-dimensional shape spaces form a Poncelet pencil of circles in the Euclidean plane with limit points ω and $\overline{\omega}$.

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ЕДНОМЕРНИ ШЕЙП ПРОСТРАНСТВА

Георги Хр. Георгиев, Радостина П. Енчева

Х. Сато въвежда затворена изпъкнала крива, съответна на един неизроден триъгълник, използвайки класове на еквивалентност от подобни триъгълници. В работата показваме, че тази крива е окръжност в модела на Лестър на двумерното шейп пространство. Доказваме също, че всички такива окръжности образуват сноп на Понселе.