# BINARY MATRICES AND SOME COMBINATORIAL APPLICATIONS IN THE THEORY OF $K$-VALUED FUNCTIONS 

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In this work, application of binary matrices in $k$-valued logic is given. Objects of our study are classes of $k$-valued functions and combinatorial problems related to them. We use proofs connected with finding the number of some classes of binary matrices. Some combinatorial identities and inequalities are obtained which would also be of independent interest.

1. Introduction. The purpose of this work is to demonstrate the application of binary matrices for obtaining some quantitative estimates for some classes of $k$-valued functions.

Binary (or Boolean, or ( 0,1 )-matrix) is a matrix whose elements are equal to zero or one.

Let $P_{n}^{k}=\left\{f: A^{n} \rightarrow A \mid A=\{0,1, \ldots, k-1\}, n \geq 1, k \geq 2\right\}$. The number of different values of the function $f$ is called range of $f$. The range of the function $f$ is denoted by $R n g(f)$, where $X_{f}=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}, f \in P_{n}^{k}$.

Let $\mu_{n}^{k}(q)$ be the number of functions from $P_{n}^{k}$ with range $q$, where $q \in\{1,2, \ldots, k\}$. For $\mu_{n}^{k}(q)$ [2] we have:

$$
\begin{equation*}
\mu_{n}^{k}(q)=\binom{k}{q} \sum_{\substack{r_{1}+r_{2}+\cdots+r_{q}=k^{n} \\ r_{i} \geq 1, i=1,2, \ldots, q}} \frac{k^{n}!}{r_{1}!r_{2}!\cdots r_{q}!}=\binom{k}{q} \sum_{i=1}^{q}(-1)^{q-i}\binom{q}{i} i^{k^{n}} \tag{1}
\end{equation*}
$$

A function $h$ is called a subfunction of $f \in P_{n}^{k}$ with respect to $M, M \subseteq X_{f}$ if $h$ is obtained from $f$ by replacing the variables of the set $M$ with constants, and we write $h \stackrel{M}{\prec} f$.

Let $M \subseteq X_{f}$ and $G$ be the set of all subfunctions of $f$ with respect to $X_{f} \backslash M$, i.e.

$$
G=G(M, f)=\left\{g \mid g{ }^{X_{f} \backslash M} \prec f\right\}
$$

The set $\operatorname{Spr}(M, f)=\bigcup_{g \in G}\{\operatorname{Rng}(g)\}$ is called spectrum of the set $M$ with respect to $f$.
For more details concerning above definitions and notation, see $[2,3,4]$.
In this work we consider the following sets of binary matrices:
$\mathcal{L}_{s \times t}(p, q)$ - the set of all $s \times t$ binary matrices which have at least one 1 in each row, and if the number of 1 's in the $j$-th column is equal to $t_{j}$, then $p \leq t_{j} \leq q, j=1,2, \ldots, t$. 118
$\mathcal{R}_{s \times t}(p)=\mathcal{L}_{s \times t}(p, p)$ - the set of all $s \times t$ binary matrices which have exactly $p$ ones in each column and at least one 1 in each row;
$\mathcal{H}_{s}(p) \subseteq \mathcal{R}_{s \times s}(p)$ - the set of all binary $s \times s$ matrices with exactly $p$ 1's in each row and each column.

The sets considered above can be applied in the theory of $k$-valued functions. For example, the following result is proved in [4]:

Theorem 1. [4] If $M \subseteq X_{f},|M|=m \neq 0$, then the number of functions $f \in P_{n}^{k}$, for which $\operatorname{Spr}(M, f)=\{a\}, \operatorname{Rng}(f)=b$, and $a \in N, b \in N, 1 \leq a \leq b \leq k$ is

$$
\begin{equation*}
\binom{k}{b}\left[\frac{\mu_{m}^{k}(a)}{\binom{k}{a}}\right]^{t}\left|\mathcal{R}_{b \times t}(a)\right| \tag{2}
\end{equation*}
$$

where $t=k^{n-m}$.
In [4], en explicit formula is found for $\left|\mathcal{R}_{b \times t}(a)\right|$; however, in this work, we will obtain this formula as a direct corollary of a more general result.
2. On the number of elements of some classes of binary matrices.

Theorem 2. If $1 \leq p \leq q \leq s$, then the number of binary matrices of the set $\mathcal{L}_{s \times t}(p, q)$ is given by the expression:

$$
\begin{equation*}
\left|\mathcal{L}_{s \times t}(p, q)\right|=\sum_{i=0}^{s-p}(-1)^{i}\binom{s}{i}\left[\sum_{j=p}^{q}\binom{s-i}{j}\right]^{t} \tag{3}
\end{equation*}
$$

Proof. The expression $R(p, q, s, t)=\left[\sum_{j=p}^{q}\binom{s}{j}\right]^{t}$ gives the number of all binary matrices of dimension $s \times t$ such that the number of 1 's in each column of these matrices is in the interval $[p, q]$.

Let $A$ be an $s \times t$ binary matrix. We say that $A$ has the property $r_{j}$, if $j$-th row of $A$ contains only zeroes, $1 \leq j \leq s$. Then obviously the number of matrices of the set $\mathcal{L}_{s \times t}(p, q)$ which possess the properties $r_{j_{1}}, r_{j_{2}}, \ldots, r_{j_{i}}, i=1,2, \ldots s$, is equal to $R(p, q, s-i, t)$. Then, using the principle of inclusion and exclusion, we get:

$$
\begin{equation*}
\left|\mathcal{L}_{s \times t}(p, q)\right|=\sum_{i=0}^{s}(-1)^{i}\binom{s}{i} R(p, q, s-i, t) \tag{4}
\end{equation*}
$$

Taking into account that for each matrix of the set $\mathcal{L}_{s \times t}(p, q)$ in each column there are at least $p$ 1's, i.e. the existence of $s-p$ rows of zeroes is impossible, then $R(p, q, r, t)=0$ with $r>s-p$, and we obtain formula (3).

As an immediate corollary of Theorem 2, setting $p=q$, we obtain the following proposition:

Corollary 1 ([4] Lemma 1). The number $\left|\mathcal{R}_{s \times t}(p)\right|$ of all $s \times t$ binary matrices, having exactly $p$ 1's in each column and having at least one 1 in each row, is given by
the following formula:

$$
\begin{equation*}
\left|\mathcal{R}_{s \times t}(p)\right|=\sum_{i=0}^{s-p}(-1)^{i}\binom{s}{i}\binom{s-i}{p}^{t} \tag{5}
\end{equation*}
$$

Consider some special subsets of the set $\mathcal{L}_{s \times t}(p, q)$ :
$\mathcal{L}_{s \times t}(p, q]$ - the set of all matrices of $\mathcal{L}_{s \times t}(p, q)$ which have exactly $q 1$ 's in at least one column;
$\mathcal{L}_{s \times t}[p, q)$ - the set of matrices of $\mathcal{L}_{s \times t}(p, q)$ which have exactly $p 1$ 's in at least one column;
$\mathcal{L}_{s \times t}[p, q]$ - the set of matrices of $\mathcal{L}_{s \times t}(p, q)$ which have exactly $q$ 1's in at least one column and exactly $p$ ones in at least one column.

Theorem 2 is also used in finding a quantitative estimate for the sets
$\mathcal{L}_{s \times t}(p, q], \mathcal{L}_{s \times t}[p, q)$, and $\mathcal{L}_{s \times t}[p, q]$.
Corollary 2. If $1 \leq p<q \leq s$, then the number of binary matrices of the set $\mathcal{L}_{s \times t}(p, q]$ is given by the equality:

$$
\begin{equation*}
\left|\mathcal{L}_{s \times t}(p, q]\right|=\sum_{i=0}^{s-p}(-1)^{i}\binom{s}{i}\left\{\left[\sum_{j=p}^{q}\binom{s-i}{j}\right]^{t}-\left[\sum_{j=p}^{q-1}\binom{s-i}{j}\right]^{t}\right\} \tag{6}
\end{equation*}
$$

Proof. We obtain the equality (6) taking into account the fact that

$$
\mathcal{L}_{s \times t}(p, q]=\mathcal{L}_{s \times t}(p, q) \backslash \mathcal{L}_{s \times t}(p, q-1) .
$$

Corollary 3. If $1 \leq p<q \leq s$, then the following equality holds true:

$$
\begin{equation*}
\left|\mathcal{L}_{s \times t}[p, q)\right|=\sum_{i=0}^{s-p}(-1)^{i}\binom{s}{i}\left\{\left[\sum_{j=p}^{q}\binom{s-i}{j}\right]^{t}-\left[\sum_{j=p+1}^{q}\binom{s-i}{j}\right]^{t}\right\} \tag{7}
\end{equation*}
$$

Proof. We obtain the equality (7) taking into account that $\mathcal{L}_{s \times t}[p, q)=\mathcal{L}_{s \times t}(p, q) \backslash \mathcal{L}_{s \times t}(p+1, q)$.

Corollary 4. If $1 \leq p \leq q \leq s$, then the following equality holds true:

$$
\begin{align*}
& \left|\mathcal{L}_{s \times t}[p, q]\right|= \\
& =\sum_{i=0}^{s-p}(-1)^{i}\binom{s}{i}\left\{\left[\sum_{j=p+1}^{q}\binom{s-i}{j}\right]^{t}+\left[\sum_{j=p}^{q-1}\binom{s-i}{j}\right]^{t}-\left[\sum_{j=p}^{q}\binom{s-i}{j}\right]^{t}\right\} \tag{8}
\end{align*}
$$

Proof. We obtain the equality (8) taking into account that

$$
\mathcal{L}_{s \times t}[p, q]=\mathcal{L}_{s \times t}[p, q) \bigcap \mathcal{L}_{s \times t}(p, q] .
$$

3. Some combinatorial identities and inequalities and their relation to the functions of $\boldsymbol{k}$-valued Logic. The assertion of Corollary 1 is useful because it gives us a direct method for obtaining some combinatorial identities. For example, the following proposition is known (see, for example, [6], Problem 3.11):

Proposition 1. The following identity holds true:

$$
\begin{equation*}
\sum_{i=0}^{s-1}(-1)^{i}\binom{s}{i}(s-i)^{s}=\sum_{i=0}^{s}(-1)^{i}\binom{s}{i}(s-i)^{s}=s! \tag{9}
\end{equation*}
$$

Proof. If we set $s=t$ and $p=1$ in the equality (5) of Corollary, we get the number $\left|\mathcal{H}_{s}(1)\right|=\left|\mathcal{R}_{s \times s}(1)\right|$ of all $s \times s$ binary matrices with exactly one 1 in each row and each column (permutation matrices), which number, as it is well-known, is $s$ ! (see, for example, [8]).

Proposition 2. The following identity holds true:

$$
\begin{equation*}
\sum_{i=0}^{s-1}(-1)^{i}\binom{s}{i}(s-i)^{t}=\sum_{\substack{\left(t_{1}, t_{2}, \ldots, t_{s}\right), t_{i}>0 \\ t_{1}+t_{2}+\cdots+t_{s}=t}} \frac{t!}{t_{1}!t_{2}!\cdots t_{s}!} \tag{10}
\end{equation*}
$$

Proof. Consider the equality (5) with $p=1$ and let $A \in \mathcal{R}_{s \times t}(1)$. Let in the $i$-th row of $A$ there will be $t_{i} 1$ 's. Since there is at least one 1 in each row, then $0<t_{i}$ $i=1,2, \ldots, s$, and since the total number of 1 's is $t$, then $t_{1}+t_{2}+\cdots+t_{s}=t$. Therefore, since there is only one 1 in each column,

$$
\begin{gathered}
=\mathcal{R}_{s \times t}(1)= \\
\sum_{\substack{\left(t_{1}, t_{2}, \ldots, t_{s}\right), t_{i}>0 \\
t_{1}+t_{2}+\cdots+t_{s}=t}}\binom{s}{t_{1}}\binom{s-t_{1}}{t_{2}}\binom{s-t_{1}-t_{2}}{t_{3}} \cdots s-t_{1}-\cdots-\binom{t_{s-1}}{t_{s}}= \\
=\sum_{\substack{\left(t_{1}, t_{2}, \ldots t_{s}\right), t_{i}>0 \\
t_{1}+t_{2}+\cdots+t_{s}=t}} \frac{t!}{t_{1}!t_{2}!\cdots t_{s}!} .
\end{gathered}
$$

Corollary 5. If $M \subseteq X_{f},|M|=m \neq 0$, then the number of functions $f \in P_{n}^{k}$, for which $\operatorname{Spr}(M, f)=\{1\}, \operatorname{Rng}(f)=b$, and $b \in N, 1 \leq b \leq k$, is

$$
\begin{equation*}
\binom{k}{b} \sum_{i=0}^{b-1}(-1)^{i}\binom{b}{i}(b-i)^{t}=\binom{k}{b} \sum_{\substack{\left(t_{1}, t_{2}, \ldots, t_{s}\right), t_{i}>0 \\ t_{1}+t_{2}+\cdots+t_{s}=t}} \frac{t!}{t_{1}!t_{2}!\cdots t_{s}!} \tag{11}
\end{equation*}
$$

where $t=k^{n-m}$.
Proof. We apply Theorem 1 and Corollary 1 with $a=1$. In this case we have $\mu_{m}^{k}(1)=\binom{k}{1}=k$. From Proposition 2, if we set $s=b$ in (10), we get (11).

It is not difficult to observe the validity of the following inclusion:

$$
\begin{equation*}
\mathcal{H}_{s}(p) \subseteq \mathcal{R}_{s \times s}(p) \tag{12}
\end{equation*}
$$

from which we obtain the following inequality:

$$
\begin{equation*}
\left|\mathcal{H}_{s}(p)\right| \leq\left|\mathcal{R}_{s \times s}(p)\right| \tag{13}
\end{equation*}
$$

where the equality occurs for $p=s$, and in this case both sets are singletons, consisting only of the $s \times s$ matrix, containing only 1's. The equality is also attained for $p=1$ (see Proposition 1).

Then from (13), applying the known formulas about the number of all $s \times s$ binary matrices with exactly $p$ 1's in each row and each column with $p=2$ and $p=3$, we get the respective inequalities. In both cases, in order to apply Theorem 1, we have to set $s=b=t=k^{n-m}$, taking into account the inequality $b \leq k$, whence $k^{n-m} \leq k$. The last inequality is valid for positive integers $n, m$ and $k \geq 2$ if and only if $n-m=0$, For $n-m=0$ we have $s=b=t=1$ and this case is out of interest for us. That is why, in our further consideration, in the estimate of the number of these functions of the class $P_{n}^{k}$ such that

$$
\begin{equation*}
m=n-1 \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
b=t=k \tag{15}
\end{equation*}
$$

Proposition 3. The following inequality holds true:

$$
\begin{equation*}
\sum_{2 t_{2}+3 t_{3}+\cdots+s t_{s}=s} \frac{(s!)^{2}}{\prod_{j \geq 2} t_{j}!(2 j)^{t_{j}}} \leq \sum_{i=0}^{s-2}(-1)^{i}\binom{s}{i}\binom{s-i}{2}^{s} \tag{16}
\end{equation*}
$$

Proof. In [7] it is shown that the number of all $s \times s$ binary matrices with exactly two 1's in each row and each column is equal to

$$
\begin{equation*}
\left|\mathcal{H}_{s}(2)\right|=\sum_{2 t_{2}+3 t_{3}+\cdots+s t_{s}=s} \frac{(s!)^{2}}{\prod_{j \geq 2} t_{j}!(2 j)^{t_{j}}} \tag{17}
\end{equation*}
$$

Then from (5) and (13) with $p=2$ we get (16).
From Theorem 1, Corollary 1 and Proposition 3, if we set $a=2$ and $s=b=k$, we get the following

Corollary 6. If $M \subseteq X_{f},|M|=n-1$, then the number of functions $f \in P_{n}^{k}$, for which $\operatorname{Spr}(M, f)=\{2\}, \operatorname{Rng}(f)=k$, and $k \in N, 2 \leq k$, is greater than

$$
\begin{equation*}
\left[\frac{\mu_{n-1}^{k}(2)}{\binom{k}{2}}\right]^{k} \sum_{2 t_{2}+3 t_{3}+\cdots+k t_{k}=k} \frac{(k!)^{2}}{\prod_{j \geq 2} t_{j}!(2 j)^{t_{j}}} \tag{18}
\end{equation*}
$$

Proposition 4. The following inequality holds true:

$$
\begin{equation*}
\frac{(s!)^{2}}{6^{s}} \sum_{\alpha+\beta+\gamma=s} \frac{(-1)^{\beta}(\beta+3 \gamma)!2^{\alpha} 3^{\beta}}{\alpha!\beta!(\gamma!)^{2} 6^{\gamma}} \leq \sum_{i=0}^{s-3}(-1)^{i}\binom{s}{i}\binom{s-i}{3}^{s} \tag{19}
\end{equation*}
$$

Proof. In [1], the following formula is given

$$
\begin{equation*}
\left|\mathcal{H}_{s}(3)\right|=\frac{(s!)^{2}}{6^{s}} \sum_{\alpha+\beta+\gamma=s} \frac{(-1)^{\beta}(\beta+3 \gamma)!2^{\alpha} 3^{\beta}}{\alpha!\beta!(\gamma!)^{2} 6^{\gamma}} \tag{20}
\end{equation*}
$$

and it is stated in [5] that (20) is "the most explicit" formula for calculating $\mathcal{H}_{s}(3)$, known by the moment of its statement. Then from (5) and (13) with $p=3$ we get (19).

From Theorem 1, Corollary 1 and Proposition 4, if we set $a=3$ and $s=b=k$, we get the following

Corollary 7. If $M \subseteq X_{f},|M|=n-1$, then the number of functions $f \in P_{n}^{k}$, for which $\operatorname{Spr}(M, f)=\{3\}, \operatorname{Rng}(f)=k$, and $k \in N, 3 \leq k$, is greater than

$$
\begin{equation*}
\left[\frac{\mu_{n-1}^{k}(3)}{\binom{k}{3}}\right]^{k} \frac{(k!)^{2}}{6^{k}} \sum_{\alpha+\beta+\gamma=k} \frac{(-1)^{\beta}(\beta+3 \gamma)!2^{\alpha} 3^{\beta}}{\alpha!\beta!(\gamma!)^{2} 6^{\gamma}} \tag{21}
\end{equation*}
$$

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## БИНАРНИ МАТРИЦИ И НЯКОЙ КОМБИНАТОРНИ ПРИЛОЖЕНИЯ В ТЕОРИЯТА НА $K$-ЗНАЧНИТЕ ФУНКЦИИ

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В работата е показано приложението на бинарните матрици в $k$-значната логика. Обект на изследване са класове от $k$-значни функции и свързани с тях комбинаторни задачи. Използувани са доказателства свързани с намиране броя на някои класове бинарни матрици. Получени са и някои комбинаторни тъждества и неравенства, които биха представлявали и самостоятелен интерес.

