# ON THE AUTOMORPHISM GROUP OF THE HYPOTHETICAL 2-(40,10,3) DESIGN* 

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In this paper, we prove that the full automorphism group of the hypothetical quasiresidual 2-(40,10,3) design is a 2 -group or the trivial group.

1. Preliminaries. We assume that the reader is familiar with the general notions and results from design theory $[1,3,6]$. In this paper, we consider the automorphisms of the hypothetical quasiresidual $2-(40,10,3)$ design. This is one of the smallest parameter sets (with respect to the number of points) for block design for which the existence question is still undecided [5]. The coresponding symetric $2-(53,13,3)$ does not exist by the Bruck-Ryser-Chowla Theorem. It has been proved by Topalova [7] that the only prime divisors of the full automorphism group of a design with the above parameters are 2 and 3. In this paper, we rule out the possibility of an automorphism of order 3.
2. Some general results. We start by a theorem related to the Fisher inequality.

Theorem 2.1. Let $\mathcal{D}$ be a $2-(v, k, \lambda)$ design with an automrphism $\sigma$ of prime order $p$, where $p \mid \lambda, p \backslash k$ and $p \backslash r$. Let further $f$ be the number of fixed points and $g$ - the number of fixed blocks of $\mathcal{D}$ under $\sigma$. Then $f \leq g$.

Proof. Denote by $T$ the part of the incidence matrix of $\mathcal{D}$ corresponding to the fixed points and the fixed blocks. Since the number of fixed blocks containing a fixed point (resp., a pair of fixed points) is $\equiv r(\bmod p)(\operatorname{resp} ., \equiv \lambda(\bmod p)), T$ satisfies the equality

$$
\begin{equation*}
T T^{t}=(r-\lambda) I+\lambda J \quad(\bmod p) \tag{1}
\end{equation*}
$$

where $I$ is the unit matrix of order $v$ and $J$ is the all-one matrix of order $v$. From (1), we get $\operatorname{det} T T^{t}=r k(r-\lambda)^{v-1}$ which implies that $\operatorname{det}\left(T T^{t}\right) \neq 0$ considered as an element of $\mathbb{F}_{p}$. Hence

$$
g \geq \operatorname{rank}_{p} T \geq \operatorname{rank}_{p} T T^{t}=f
$$

Let $A$ be the (point-by-block) incidence matrix of a $2-(v, k, \lambda)$ design. The intersection matrix $S=A^{t} A=\left(s_{i j}\right)$ is defined as the matrix which has in its $(i, j)$-th position the number of common points of the $i$-th and $j$-th block. The following inequality is due to Connor [2].

Theorem 2.2. [2]

$$
-(r-\lambda-k) \leq s_{i j} \leq \frac{2 \lambda k+r(r-\lambda-k)}{r} .
$$

[^0]Further, we make use of a simple observation by Connor [2], used also by Hall, Roth, van Rees and Vanstone [4] in the investigation of the parameters $2-(22,8,4)$. It allows us to prove restrictions on the intersection numbers of a given design $\mathcal{D}$. Again, let $A$ be the incidence matrix of the $2-(v, k, \lambda)$ design $\mathcal{D}$ and let $S$ be its intersection matrix. Clearly,

$$
S^{2}=A^{t}\left(A A^{t}\right) A^{t}=A^{t}((r-\lambda) I+\lambda J) A=(r-\lambda) S+\lambda k^{2} J .
$$

The projection matrix $C$ of $\mathcal{D}$ is defined by

$$
C=r(r-\lambda) I+\lambda k J-r S
$$

It is easily checked that $C^{2}=r(r-\lambda) C$. Therefore, $C$ corresponds to a positive semidefinite quadratic form and no principal minor of $C$ can have a negative determinant.
3. The hypothetical $2-(40,10,3)$ designs. Let $\mathcal{D}$ be a $2-(40,10,3)$ design. Applying the results from the previous section, we obtain the following properties of such designs:
(A) two blocks of $\mathcal{D}$ meet in at most 4 points, i.e. $s_{i j} \leq 4$;
(B) there cannot exist three blocks $B_{i}, i=1,2,3$, in $\mathcal{D}$ with $\left|B_{i} \cap B_{j}\right|=4$, for any pair $(i, j)$;
(C) two fixed blocks intersect in 0 or 3 fixed points.

For the intersection numbers, one gets the system:

$$
\sum_{i=0}^{4} n_{i}=51, \quad \sum_{i=1}^{4} i n_{i}=120, \quad \sum_{i=2}^{4} i n_{i}=90
$$

which has the following solutions:

| $n_{0}$ | $n_{1}$ | $n_{2}$ | $n_{3}$ | $n_{4}$ |
| ---: | ---: | ---: | ---: | ---: |
| 1 | 0 | 30 | 20 | 0 |
| 0 | 3 | 27 | 21 | 0 |
| 0 | 2 | 30 | 18 | 1 |
| 0 | 1 | 33 | 15 | 2 |
| 0 | 0 | 36 | 12 | 3 |

Let $\sigma$ be an automorphism of order 3 of $\mathcal{D}$. Denote by $s$ (resp. $t$ ) the number of nontrivial point (resp. block) orbits under $\sigma$. Obviously $f+3 s=40$ and $g+3 t=52$. By Theorem 1, we have $f \leq g$.

Denote by $M=\left(m_{i j}\right)$ the orbit matrix of $\mathcal{D}$ with respect to $\sigma$. Let $t_{i}$ be the number of fixed blocks containing the points of the $i$-th point orbit; similarly, let $t_{c d}$ be the number of fixed blocks containing the point orbits with numbers $c$ and $d$. The following identities are proved by a straightforward counting:

$$
\begin{align*}
\sum_{j} m_{i j} & =r-t_{i},  \tag{3}\\
\sum_{j} m_{i j}\left(m_{i j}-1\right) & =(p-1)\left(\lambda-t_{i}\right), \\
\sum_{j} m_{c j} m_{d j} & =p\left(\lambda-t_{c d}\right), c \neq d .
\end{align*}
$$

Theorem 3.1. $g \leq 10$.
Proof. The orbit matrix $M$ can be represented in the form

$$
M=\left(\begin{array}{cc}
T & U \\
V & W
\end{array}\right)
$$

where $T$ corresponds to the fixed points and blocks; $U$ - to the fixed point and nontrivial block orbits; $V$ - to the nontrivial point orbits and the fixed blocks and $W$ - to the nontrivial point and block orbits. By (3-5), a row in $W$ has one of the following five forms (up to a permutation):

$$
\begin{gathered}
(3,1,1,1,1,1,1,1,1,1,1,0, \ldots, 0),(2,2,2,1,1,1,1,1,1,1,0, \ldots, 0), \\
(2,2, \underbrace{1, \ldots, 1}_{8}, 0, \ldots, 0),(2, \underbrace{1, \ldots, 1}_{9}, 0, \ldots, 0),(\underbrace{1, \ldots, 1}_{10}, 0, \ldots, 0) .
\end{gathered}
$$

Assume $g \geq 16$, i.e. $t \leq 12$. The scalar product of two rows of $W$ is $>6$ and, therefore, no fixed block contains two nontrivial point orbits. Count the triples $(P, Q, B)$, where $P$ and $Q$ are fixed points and $B$ is a fixed block that contains $P$ and $Q$, one gets

$$
\binom{f}{2} \cdot 3 \geq 16\binom{7}{2}
$$

i.e. $f \geq 16$. By Theorem 2.1, $f=16$.

From $f=g$ we get that any two fixed blocks meet in 0 or 3 fixed points. Hence, there exist two possibilities:
(a) any fixed block contains 7 fixed points
(b) 15 fixed blocks contain 7 fixed points and one fixed block has 10 fixed points.

Assume that the fixed point $P$ is in 10 fixed blocks. Then, the number of the triples $(P, Q, B)$, where $Q$ is a fixed point, $B$ is a fixed block and $P, Q \in B$ is $\geq 10 \cdot 6=60$. Hence $f \geq 1+(60 / 3)=21$, a contradiction. Therefore, each fixed point is in $\leq 7$ fixed blocks and case (b) is impossible.

Furthermore, note that exactly 8 pairs of fixed points do not appear in fixed blocks. Now case $(a)$ is ruled out by counting the number of nontrivial block orbits containig fixed points. Their number is $t \geq 16 \cdot 2-8=24$, a contradiction.

Now, let $g=13$ and count as above the triples $(P, Q, B)$, where $P$ and $Q$ are fixed points, $B$ is a fixed block and $P, Q \in B$. As before, a fixed block cannot contain two nontrivial point orbits. Hence, $\binom{f}{2} \cdot 3 \geq\binom{ 7}{2} \cdot 13$, i.e. $f>13$, a contradiction to Theorem 2.1.

Theorem 3.2. The case $f=10, g=10$ is impossible.

Proof. For a given fixed point $x$, denote by $r_{x}$ the number of fixed blocks containing $x$. Obviously, $r_{x} \equiv 1(\bmod 3)$. It is impossible to have $r_{x}=10$, since then $r_{y}=1$ for all other fixed points. It is impossible to have also a fixed point $x$ with $r_{x}=7$. Since every two fixed blocks intersect in 0 or 3 fixed points, the intersection of two fixed blocks containing $x$ is exactly 3 . Then the matrix

$$
\left(\begin{array}{rrrrrrr}
30 & -9 & -9 & -9 & -9 & -9 & -9 \\
-9 & 30 & -9 & -9 & -9 & -9 & -9 \\
-9 & -9 & 30 & -9 & -9 & -9 & -9 \\
-9 & -9 & -9 & 30 & -9 & -9 & -9 \\
-9 & -9 & -9 & -9 & 30 & -9 & -9 \\
-9 & -9 & -9 & -9 & -9 & 30 & -9 \\
-9 & -9 & -9 & -9 & -9 & -9 & 30
\end{array}\right)
$$

is a principal minor of $C$ and has a negative determinant. If there is a point $x$ with $r_{x}$, we can prove that there exist five fixed blocks every two of them meeting in three points. Again, this gives principal minor of negative determinant. Thus, every fixed point appears in exactly one fixed block and $T=I_{10}$.

By the fact that the Ramsey number $K(3,4)=9$ there exist either four nontrivial point orbits every two of which meet in a fixed block, or three such orbits no two of which meet in the fixed part. The first alternative is clearly impossible since the corresponding four rows do not have a common zero position in the nonfixed part. For the second alternative, there exist four rows in $W$ that are equivalent to one of

$$
\begin{aligned}
& \left(\begin{array}{llllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\right), \\
& \left(\begin{array}{lllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0
\end{array}\right) .
\end{aligned}
$$

Now, consider a point orbit which is contained in the unique fixed block not containing the three disjoint point orbits. The corresponding row in $W$ has scalar product 9 with the first two rows and 6 with the third. This is clearly impossible.

Theorem 3.3. If $f=1$ then $g \leq 4$, i.e. the cases $f=1, g=7$ and $f=1, g=10$ are impossible.

Proof. Every two fixed blocks intersect in 1 or 4 points. Hence the intersection numbers for any fixed block satisfy $n_{1}+n_{4} \geq 6$, a contradiction to (2).

Theorem 3.4. The cases $f=7, g=10$ and $f=4, g=10$ are impossible.
Proof. We are going to sketch the proof for $f=7, g=10$. The case of $f=4, g=10$ is proved analogously. It is readily seen that all fixed contain exactly one fixed point. Moreover, one of the fixed points, say $a$, is contained in four fixed blocks while the remaining fixed points are contained in exactly one fixed block each. Let $B_{1}, \ldots, B_{4}$ be the fixed blocks containing $a$ and $B_{6}, \ldots, B_{10}$ be the remaining fixed blocks. For every $1 \leq i \leq 4$ and every $5 \leq j \leq 10$, we have $\left|B_{i} \cap B_{j}\right|=3$ since it is impossible to have $n_{0}+n_{1}+n_{4} \geq 4$ for some block, cf. (2). Now $\left|B_{i} \cap B_{j}\right|=1$ for every $1 \leq i<j \leq 4$. The number of nontrivial point orbits in the blocks $B_{1}, \ldots, B_{4}$ is 12 , a contradiction to $f=7$.

For designs with $f, g \leq 7$, all the possibilities for the matrices $T, U$ and $V$ are easily constructed. It turns out (computer check) that they cannot be extended to a matrix that satisfies equations (3-5). Consequently, designs with parameters 2-(40,10,3) and an automorphism of order 3 do not exist. Thus we have the following result.

Theorem 3.5. There exist no $2-(40,10,3)$ designs with an automorphism of order 3.
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# ВЪРХУ ГРУПАТА ОТ АВТОМОРФИЗМИ НА ДИЗАЙНИТЕ С ПАРАМЕТРИ 2-(40,10,30) 

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В тази статия доказваме, че пълната група от автоморфизми на хипотетичния квазиоостатъчен $2-(40,10,3)$ дизайн е 2 -група или тривиалната група.


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