# RUNGE-KUTTA METHODS FOR AGE-STRUCTURED POPULATION EQUATIONS 

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We present Lotka-McKenrick's model which describes the evolution in time of the age structure of a population. An alternative way for solving it in terms of the Runge-Kutta methods applied on the Renewal equation and the equation with the Age profile is discussed and we show the connection between Lotka's model and these two equations. Our approaches are based on a biologically significant case, i.e. when the species are with a finite life-span, which creates additional difficulties for the numerical treatment of the problem.

1. Lotka-McKendrick's model, Renewal equation and the equation with the Age profile. Let us consider the linear Lotka-McKendrick's model (known also as McKendrick-von Foerster's), i.e. we have a single population; individuals are neither with sex differences, nor dependent on their size, but they are structured by age:

$$
\left\{\begin{array}{l}
\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}+\mu(a) p=0, \quad a \in\left[0, a_{+}\right], t>0  \tag{1.1}\\
p(0, t)=\int_{0}^{a_{+}} \beta(a) p(a, t) \mathrm{d} a=B(t), \quad t>0 \\
p(a, 0)=p_{0}(a), \quad a>0
\end{array}\right.
$$

We have used the following notations:
$\mathbf{p}(\mathbf{a}, \mathbf{t})$ - the age density of the population, where $a \in\left[0, a_{+}\right]$and $t \geq 0 \quad\left(a_{+}\right.$is the maximum age).
$p(a, 0)$ - the initial age distribution of the population.
$\beta(a)$ - age specific fertility.
$\mu(\mathbf{a})$ - age specific mortality.
$\boldsymbol{\pi}(\mathbf{a})=\mathbf{e}^{-\int_{0}^{\mathbf{a}} \mu(\tau) \mathbf{d} \tau}$ - survival probability.
$\mathbf{B}(\mathrm{t})$ - the birth rate i.e. the number of offspring in one time unit.
In order to allow the mathematical treatment of (1.1), we need to specify some more conditions:

- we want $a_{+}<+\infty$;
- we consider the total birth rate $B(t)=\int_{0}^{a_{+}} \beta(a) p(a, t) d a$;
- we want $\pi(a)=e^{-\int_{0}^{a} \mu(\tau) d \tau}$ vanishes at $a_{+}$, i.e. $\pi\left(a_{+}\right)=0$;
$-p_{0} \in L^{1}\left(0, a_{+}\right), p_{0}(a) \geq 0$ in $\left[0, a_{+}\right]$;
$-\int_{0}^{a_{+}} \mu(\tau) d \tau=+\infty\left(\right.$ it is necessary for $\left.\pi\left(a_{+}\right)=0\right)$

Integrating the governing equation in (1.1) along the characteristic lines (see [1]), we have:

$$
p(a, t)=\left\{\begin{array}{l}
p_{0}(a-t) \frac{\pi(a)}{\pi(a-t)}, \quad a \geq t  \tag{1.2}\\
B(t-a) \pi(a), \quad a<t
\end{array}\right.
$$

It can be shown (see [1]) that problem (1.1) is equivalent to the following Volterra integral equation of second kind (Renewal equation):

$$
B(t)=\left\{\begin{array}{l}
F(t)+\int_{0}^{t} K(t-a) B(a) d a, t \leq a_{+}  \tag{1.3}\\
\int_{t-a_{+}}^{t} K(t-a) B(a) d a, t>a_{+}
\end{array}\right.
$$

where

$$
\begin{align*}
& F(t)=\int_{t}^{a_{+}} \beta(a) p_{0}(a-t) \frac{\pi(a)}{\pi(a-t)} d a, \quad t \leq a_{+} \\
& F(t)=0, t \geq a_{+}  \tag{1.4}\\
& K(a)=\beta(a) \pi(a)
\end{align*}
$$

Once we have the solution of the Renewal equation, i.e. the value of $B(t)$ for $t \in[0, T]$ and substituting it in (1.2), we can obtain the solution of the previous problem.
Another way to look at the solution of Lotka-McKendrick's equation is via the equation with the Age profile. Let us consider the following variables:

$$
\left\{\begin{array}{l}
w(a, t)=\frac{p(a, t)}{P(t)} \text { (age profile) }  \tag{1.5}\\
P(t)=\int_{0}^{a_{+}} p(a, t) d a \text { (total population) }
\end{array}\right.
$$

By the definition of $\mathrm{w}(\mathrm{a}, \mathrm{t})$ and $\mathrm{P}(\mathrm{t})$ itself and by differentiating the expression $p(a, t)=$ $w(a, t) P(t)$ and then substituting in (1.1) (see [1]), we get the following equations:

$$
\left\{\begin{array}{l}
w_{t}(a, t)+w_{a}(a, t)+\mu(a) w(a, t)+\alpha(t) w(a, t)=0  \tag{1.6}\\
w(0, t)=\int_{0}^{a} \beta(a) w(a, t) d a \\
\int_{0}^{a_{+}} w(a, t) d a=1 \\
w(a, 0)=w_{0}(a)
\end{array}\right.
$$

$$
\left\{\begin{array}{l}
\frac{d}{d t} P(t)=\alpha(t) P(t)  \tag{1.7}\\
P(0)=P_{0}
\end{array}\right.
$$

where:

$$
\begin{equation*}
w_{0}(a)=\frac{p_{0}(a)}{\int_{0}^{a_{+}} p_{0}(\tau) d \tau}, \quad P_{0}=\int_{0}^{a_{+}} p_{0}(\tau) d \tau \tag{1.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha(t)=\int_{0}^{a_{+}}[\beta(\tau)-\mu(\tau)] w(\tau, t) d \tau \tag{1.9}
\end{equation*}
$$

When we obtain the solution of (1.6), we can find the solution of the former initialboundary value problem by using formulas (1.5), (1.7) and (1.8). In that case the leading equation in the model above is much more complicated (it is an hyperbolic, nonlinear equation) than the Lotka-McKendrick's one and the Renewal equation, because here are nonlinearities to deal with. But the most useful property of (1.6) is that its solution is bounded (see [1]) and it follows that the numerical schemes developed for it will have better long time behavior.
2. Numerical schemes. During the last years many direct methods for McKen-drick-von Foerster's equation have been proposed (see [4], [5], [6]), but no approaches via the Renewal equation and the equation with the Age profile have been demonstrated. In this section we will propose Runge-Kutta schemes for these equations.

Let $h>0$ be the discretization step and $h=\frac{a_{+}}{N}$, where N is the total number of subintervals (we assume that the mesh size in time and in age is equal), i.e. we have $\left\{\left(a_{i}, t^{n}\right): x_{i}=i h, i=0, \ldots M ; t^{n}=n h, n=0, \ldots N\right\}$.
a) A forth order Runge-Kutta scheme for the Renewal equation. Let us have a Volterra integral equation of second kind and let we present it in the following form:

$$
\begin{equation*}
y(t)=F(t)+\int_{0}^{t_{n}} K(t, s) y(s) d s+\int_{t_{n}}^{t} K(t, s) y(s) d s=F^{n}(t)+\int_{t_{n}}^{t} K(t, s) y(s) d s \tag{2.1}
\end{equation*}
$$

Note: In our concrete case, the kernel $K(t, s)=K(t-s)$.
In the following we present the only existing explicit, 4 -stage and 4 -th order RK formula of Pouzet type (see [2]) (it is analogues to the forth order one that is most used for ODE's) applied on the equation (1.3):

$$
\begin{align*}
& Y_{1}^{n}=\widetilde{F}^{n}\left(t_{n}\right) \\
& Y_{2}^{n}=\widetilde{F}^{n}\left(t_{n}+\frac{h}{2}\right)+\frac{h}{2} K\left(\frac{h}{2}\right) Y_{1}^{n} \\
& Y_{3}^{n}=\widetilde{F}^{n}\left(t_{n}+\frac{h}{2}\right)+\frac{h}{2} K(0) Y_{2}^{n}  \tag{2.2}\\
& Y_{4}^{n}=\widetilde{F}^{n}\left(t_{n}+h\right)+h K\left(\frac{h}{2}\right) Y_{3}^{n} \\
& B^{n+1}=\widetilde{F}^{n}\left(t_{n}+h\right)+\frac{h}{6}\left[K(h) Y_{1}^{n}+2 K\left(\frac{h}{2}\right) Y_{2}^{n}+2 K\left(\frac{h}{2}\right) Y_{3}^{n}+K(0) Y_{4}^{n}\right]
\end{align*}
$$

where $\widetilde{F}^{n}\left(t_{n}\right)$ is an approximation of the lag term $F^{n}(t)$.
b) A second order Runge-Kutta scheme for the equation with the Age profile. Here we consider equation (1.6) and we approximate it with a second order method. This implies that for the approximation of the integral terms we have to use a second order method (for example the trapezoidal rule which is a second order accurate). It requires an evaluation of the integrated function at the right endpoint $a_{+}$of the interval. This represents a problem for the model (1.6) since $\lim _{a \rightarrow a_{+}} \mu(a)=\infty$. To avoid
this problem we make the substitution:

$$
\begin{equation*}
v(a, t)=\pi^{-1}(a) w(a, t) \tag{2.3}
\end{equation*}
$$

and we assume that:

$$
\begin{equation*}
\sup _{a \in\left[0, a_{+}\right]} \mu(a) \pi(a) \leq \mu^{*}<\infty \tag{2.4}
\end{equation*}
$$

After the substitution (2.3), (1.6) transforms into:

$$
\begin{cases}1) & v_{t}(a, t)+v_{a}(a, t)=-v(a, t) A(t)  \tag{2.5}\\ 2) & v(0, t)=\int_{0}^{a_{+}} \beta(a) \pi(a) v(a, t) d a \\ 3) & \int_{0}^{a_{+}} \pi(a) v(a, t) d a=1 \\ 4) & v(a, 0)=\pi^{-1}(a) w_{0}(a)=v_{0}(a)\end{cases}
$$

where, for simplicity, we have denoted $\mathrm{A}(\mathrm{t})=\int_{0}^{a_{+}}[\beta(\tau)-\mu(\tau)] \pi(\tau) v(\tau, t) d \tau$.
Let $V_{i}^{n}$ be an approximation of $v\left(a_{i}, t^{n}\right)$. Then, we propose an explicit second order RK scheme combined with the use of trapezoidal rule and midpoint rule as follows:

$$
\left\{\begin{array}{l}
V_{i+1}^{n+1}=V_{i}^{n}+K_{2}, \quad i=0, \ldots N-1 ; n \geq 0  \tag{2.6}\\
K_{1}=-h A^{n} V_{i}^{n}, \quad i, n \geq 0 \\
K_{2}=-A^{n+\frac{1}{2}}\left(V_{i}^{n}+\frac{K_{1}}{2}\right), \quad i, n \geq 0
\end{array}\right.
$$

By these formulas we find the solution for the new time level $t^{n+1}$ at the grid points $a_{1} \ldots a_{N}$. For the boundary points we apply the trapezoidal rule and thus we obtain:
(2.7) $V_{0}^{n+1}=\frac{h}{\left(2-h \beta_{0} \pi_{0}\right)}\left[2 \beta_{1} \pi_{1} V_{1}^{n+1}+2 \beta_{2} \pi_{2} V_{2}^{n+1}+\cdots+2 \beta_{N-1} \pi_{N-1} V_{N-1}^{n+1}+\beta_{N} \pi_{N} V_{N}^{n+1}\right]$

As it can be seen, the use of this scheme is not trivial, because in order to find $V_{i+1}^{n+1}$ first we need to find $A^{n}$ and $A^{n+\frac{1}{2}}$ which are unknown values. So, we add the following complimentary condition:

$$
\begin{equation*}
A^{n}=\frac{h}{2}\left[\left(\beta_{0}-\mu_{0}\right) \pi_{0} V_{0}^{n}+2 \sum_{i=1}^{N-1}\left(\beta_{i}-\mu_{i}\right) \pi_{i} V_{i}^{n}+\left(\beta_{N}-\mu_{N}\right) \pi_{N} V_{N}^{n}\right] \tag{2.8}
\end{equation*}
$$

where we have assumed that $\mu_{N} \pi_{N}$ is a finite number.
If we take a look at the second multiplier in the third equation in (2.6), we can notice that it is in fact an approximation of our solution for time $\left(t^{n}+\frac{h}{2}\right)$ found by making a half step of Euler's method for ODEs. It implies that we know all "inner" points for time level $\left(t^{n}+\frac{h}{2}\right)$. Then, we can use the midpoint rule in order to calculate our integral:

$$
\begin{equation*}
A^{n+\frac{1}{2}}=h \sum_{i=0}^{N-1}\left(V_{i}^{n}+\frac{K_{1}}{2}\right)\left(\beta_{i+\frac{1}{2}}-\mu_{i+\frac{1}{2}}\right) \pi_{i+\frac{1}{2}} \tag{2.9}
\end{equation*}
$$

Now putting together $(2.6),(2.7),(2.8)$ and (2.9) we complete the method.
The particular thing here is the way we calculate $A^{n+\frac{1}{2}}$ - by using another quadrature formula of the same order which is much better than using extrapolations. More general approaches to Gurtin-MacCamy's (a generalization of the linear Lotka-McKendrick's one) 176
model based on RK methods of different order $(\geq 2)$ are done in [3]. But for finding the values of $A\left(t^{n}+c_{i} h\right)$ they have done first extrapolations followed by iterations which increases the computational time and cost.
3. Discussion and results. Test example: We assume the maximum age $a_{+}=1$; we take the fertility $\beta(a)=2$; the mortality $\mu(a)=\frac{1}{1-a}$ and it follows that the survival probability $\pi(a)=1-a$. The initial values are chosen in such a way that some compatibility conditions are satisfied in order to obtain continuity of the solution (see [4]):

$$
p_{0}(a)=\left\{\begin{array}{l}
(1-2 a)^{3}(1-a), a \in\left[0, \frac{1}{2}\right]  \tag{3.1}\\
31(2 a-1)^{3}(1-a), a \in\left[\frac{1}{2}, 1\right]
\end{array}\right.
$$

Considering (2.3) and formula (1.8), we can calculate $v_{0}(a)$ for the profile:

$$
v_{0}(a)=\left\{\begin{array}{l}
2(1-2 a)^{3}, a \in\left[0, \frac{1}{2}\right]  \tag{3.2}\\
62(2 a-1)^{3}, a \in\left[\frac{1}{2}, 1\right]
\end{array}\right.
$$

$$
\begin{align*}
& K(a)=2(1-a) \\
& F(t)=2 \int_{t}^{1} p_{0}(a-t) d a, t \in[0,1] ;(F(t)=0, t>1) \tag{3.3}
\end{align*}
$$

Substituting this data into the integral equation (1.3) and differentiating it in $t$, we obtained a differential equation on $B(t)$ for the first interval $t \in[0,1]$ and a differential delay equation for $B(t)$ in $t \geq 1$. We have developed a solver for delay equations in Mathematica (ask the authors for details) and then having the values for $B(t)$, we have found $p(a, t)$ by (1.2) and $w(a, t)$ by (1.5).
We have computed the effective order of convergence of the schemes by the formula:

$$
\begin{equation*}
\alpha=\frac{\ln \left(\frac{E_{h}}{E_{\frac{h}{2}}}\right)}{\ln (2)} \tag{3.4}
\end{equation*}
$$

where $E_{h}$ is the approximation error.
In the table below we have listed some results as follows: in the first column are given the values of the the discretization step; in the next two - the approximate effective order of convergence of the discussed RK methods for the Renewal equation and the equation

Table 1

| $h$ | $\alpha\left(R K_{\text {Renewal }}\right) \approx$ | $\alpha\left(R K_{\text {Age pr. }}\right) \approx$ | $E_{h}\left(R K_{\text {Renewal }}\right) \approx$ | $E_{h}\left(R K_{\text {Age pr. }}\right) \approx$ |
| :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{60}$ | 1.92 | 1.88 | $1.72 \mathrm{E}-06$ | $3.68 \mathrm{E}-02$ |
| $\frac{1}{120}$ | 2.94 | 1.94 | $2.36 \mathrm{E}-07$ | $9.98 \mathrm{E}-03$ |
| $\frac{1}{150}$ | 2.95 | 1.95 | $1.23 \mathrm{E}-07$ | $6.49 \mathrm{E}-03$ |

with the Age profile respectively; in the last two - the absolute error of these methods.

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# МЕТОДИ РУНГЕ-КУТА ЗА СТРУКТУРИРАНИ СПОРЕД ВЪЗРАСТТА ПОПУЛАЦИОННИ МОДЕЛИ 

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Разгледан е моделът на Лотка-МакКендрик, описващ еволюцията на популация, структурирана според възрастта. Дискутиран е алтернативен начин за числено решаване на този проблем посредством метода на Рунге-Кутта за "Renewal equation" и "equation with the Age profile". Посочена е връзката между модела на Лотка-МакКендрик и тези две уравнения. Засегнат е биологически значим случай, т.е. когато индивидите имат максимална възраст, която и крайно число и това е източник на допълнителни трудности при числената интерпретация на проблема.

