# COMPUTATION OF PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS BY THE OPERATIONAL CALCULUS APPROACH* 


#### Abstract

Margarita N. Spiridonova The operational calculus of Heaviside-Mikusiński [1] is used successfully for solving the initial value problem (IVP) for some types of differential equations reducing them to algebraic equations. An extention of the Heaviside algorithm for obtainig of periodic solutions of linear ordinary differential equations (LODE) with constant coefficients (CC) and of systems of such equations is presented. Using the computer algebra system (CAS) Mathematica, the periodic solutions are obtained in closed form.


1. Heaviside algorithm for solving IVP for LODE with CC. The main idea of the Operational Calculus (OC) of Oliver Heaviside was the conversion of differential equations in algebraic equations by treating the differentiation operator as an algebraic object [1]. This idea was inspired by physical considerations and Heaviside did not establish a sound mathematical theory. The first justification of his approach was done by means of the Laplace transform. Later J. Mikusiński [2] developed a direct algebraic approach to the Heaviside OC.

Mikusiński started from the classical Duhamel convolution

$$
\begin{equation*}
(f * g)=\int_{0}^{t} f(t-\tau) g(\tau) d \tau \tag{1}
\end{equation*}
$$

considering the space $C[0, \infty)$ of the continuous functions on $[0, \infty)$ as a ring on $\mathbb{R}$ or $\mathbb{C}$. Further, he used the classical Titchmarsh theorem that the operation (1) has no divisors of zero. In the same way, as the ring $\mathbb{Z}$ of the integers is extended to the field $\mathbb{Q}$ of the rational numbers, Mikusiński extended the ring $(C[0, \infty), *)$ to the smallest field $\mathcal{M}$ containing the initial ring. We name and denote it by $\mathcal{M}$, as Mikusiński's field.

The basic operator in the Mikusiński approach is the integration operator

$$
\begin{equation*}
l f(t)=\int_{0}^{t} f(\tau) d \tau \tag{2}
\end{equation*}
$$

In fact, $l$ is the convolution operator $l=\{1\} *$. The elements of $\mathcal{M}$ are convolution fractions

$$
\frac{f}{g}=\frac{\{f(t)\}}{\{g(t)\}}
$$

[^0]Further, the algebraic analogon of the differentiation operator $D=\frac{d}{d t}$ is the convolution fraction

$$
\begin{equation*}
s=\frac{1}{l} . \tag{3}
\end{equation*}
$$

The basic formula of the Heaviside-Mikusiński OC is given by

$$
\begin{equation*}
\left\{f^{\prime}(t)\right\}=s\{f(t)\}-f(0) \tag{4}
\end{equation*}
$$

when $f \in C^{1}[0, \infty)$ and where $f(0)$ is considered as a "numerical operator".
Let us consider how the Mikusiński's approach is applied for solution of IVP for ordinary LDE with constant coefficients.

Let $P(\lambda)=a_{0} \lambda^{n}+a_{1} \lambda^{n-1}+\cdots+a_{n-1} \lambda+a_{n}$ be a non-zero polynomial of degree $n=\operatorname{deg} P$. Consider the following linear differential equation with given initial values:

$$
\begin{equation*}
P\left(\frac{d}{d t}\right) y=f(t), y(0)=\alpha_{0}, y^{\prime}(0)=\alpha_{1}, \cdots, y^{(n-1)}(0)=\alpha_{n-1} \tag{5}
\end{equation*}
$$

Using formula (4), the problem (5) is reduced to the following single algebraic equation of $1^{\text {st }}$ degree:

$$
P(s) y=f+Q(s) \quad \text { with } \quad Q(\lambda)=\sum_{j=1}^{n}\left(\sum_{k=j}^{n} a_{n-j} \alpha_{k-j}\right) s^{j-1} .
$$

Here $\operatorname{deg} Q<\operatorname{deg} P$.
The formal solution of the above equation has the form

$$
\begin{equation*}
y=\frac{1}{P(s)} f+\frac{Q(s)}{P(s)} . \tag{6}
\end{equation*}
$$

Further, we can decompose $1 / P(s)$ and $Q(s) / P(s)$ in elementary fractions and these fractions can be interpreted as functions using formulae such as:

$$
\frac{1}{(s-a)^{n}}=\left\{\frac{t^{n-1}}{(n-1)!} e^{a t}\right\}, n=1,2, \ldots
$$

After applying these formulae, we obtain the functions
$1 / P(s)=G(t), Q(s) / P(s)=H(t)$
thus obtaining the solution in the form $y=G(t) * f(t)+H(t)$.

## Example 1.

$y^{(4)}(t)-5 y^{\prime \prime}(t)+4 y(t)=t^{2}$
$y(0)=0, y^{\prime}(0)=0, y^{\prime \prime}(0)=0, y^{(3)}(0)=0$
The algebraization process leads to: $\left(4-5 s^{2}+s^{4}\right) y(t)=t^{2}$
After performing the considered operations, we get the following solution: $y(t)=\frac{1}{24}\left(15+6 t^{2}-16 \cosh (t)+\cosh (2 t)\right)$.

All computations are made using the CAS Mathematica.
2. Extension of the Heaviside algorithm to a class of BVP for LODE with
CC. Let us consider the BVP

$$
\begin{align*}
& P\left(\frac{d}{d t}\right) y=f(t), \quad 0 \leq t \leq T \\
& \int_{0}^{T} y(\tau) d \tau=\alpha_{0}, y^{(k)}(T)-y^{(k)}(0)=\alpha_{k+1}, k=0,1, \ldots n-2 . \tag{7}
\end{align*}
$$

Dimovski and Grozdev $[5,6]$ consider ( 7 ) with $T=1$ not for its own sake, but as an intermediate step in obtaining of periodic solutions of LODE with CC, which is reduced to the problem

$$
\begin{align*}
& P\left(\frac{d}{d t}\right) y=f(t)  \tag{8}\\
& y^{(k)}(T)-y^{(k)}(0)=0, k=0,1, \ldots n-1
\end{align*}
$$

for the period interval $[0, T]$ of the periodic function $f(t)$.
The extension of the Heaviside algorithm, used here, is proposed by Dimovski and Grozdev [3-6]. The following BVP is considered:

$$
\begin{align*}
& P\left(\frac{d}{d t}\right) y=f(t), \quad 0 \leq t \leq T \\
& \int_{0}^{T} y(\tau) d \tau=\alpha_{0}=\frac{1}{T a_{n}} \int_{0}^{T} f(\tau) d \tau  \tag{9}\\
& y^{(k)}(T)-y^{(k)}(0)=0, k=0,1, \ldots n-2
\end{align*}
$$

assuming that $a_{n} \neq 0$.
Instead of the convolution (1), we use the convolution

$$
\begin{gather*}
(f * g)(t)=\frac{f(t)}{T} \int_{0}^{T} g(\tau) d \tau+\frac{g(t)}{T} \int_{0}^{T} f(\tau) d \tau  \tag{10}\\
-\frac{1}{T} \int_{0}^{t} f(t-\tau) g(\tau) d \tau-\frac{1}{T} \int_{t}^{T} f(t+T-\tau) g(\tau) d \tau
\end{gather*}
$$

and the right inverse operator $l$ of $d / d t$ :

$$
\begin{equation*}
l f(t)=\int_{0}^{t} f(\tau) d \tau-\frac{1}{T} \int_{0}^{T}(T-\tau) f(\tau) d \tau \tag{11}
\end{equation*}
$$

It is the convolution operator

$$
l=\left\{t-\frac{T}{2}\right\} *
$$

Further, convolution fractions of the form $f / g$ with $f, g \in C[0, T]$, in the case when $g$ is not a divisor of 0 of operation (10), are considered.

The algebraic analogon of $d / d t$ is the convolution fraction

$$
s=\frac{1}{l}
$$

and the basic formula of the corresponding OC is

$$
\left\{f^{\prime}(t)\right\}=s\{f(t)\}-\frac{1}{T} \int_{0}^{T} f(\tau) d \tau
$$

Here $\frac{1}{T} \int_{0}^{T} f(\tau) d \tau$ is considered as a constant function.
As in the previous section, the problem (9) reduces to $P(s) y=f+Q(s)$.
The solution can be obtained in the form

$$
y=\frac{1}{P(s)} f+\frac{Q(s)}{P(s)}, \operatorname{deg} Q \leq \operatorname{deg} P
$$

provided $P(s)$ is not a divisor of 0 . After decomposing of $1 / P(s)$ and $Q(s) / P(s)$ in elementary fractions, the following formulae are to be used for interpretation of the "algebraic solution" [5]:

$$
\begin{gathered}
\frac{1}{(S-\lambda)^{m}}=\left\{\frac{(-1)^{m}}{\lambda^{m}}+\frac{T}{(m-1)!} \frac{\partial^{m-1}}{\partial \lambda^{m-1}}\left(\frac{e^{\lambda t}}{e^{\lambda t}-1}\right)\right\}, m=1,2, \ldots \\
\frac{1}{S^{2}+\lambda^{2}}=\left\{\frac{1}{\lambda^{2}}-T \frac{\cos \lambda\left(t-\frac{T}{2}\right)}{2 \lambda \sin \lambda \frac{T}{2}}\right\}
\end{gathered}
$$

Example 2. The computations are made using Mathematica again.
$y^{\prime \prime}(t)+\pi^{2} y(t)=\sin (2 \pi t) T=1, \alpha_{0}=\frac{1}{\pi^{2}} \int_{0}^{1} \sin (2 \pi t) d t$
The algebraization process leads to: $\left(\pi^{2}+s^{2}\right) y=\sin (2 \pi t)$
The received closed form solution is: $-\sin (2 \pi t) /\left(3 \pi^{2}\right)$
Comments. For another approach to periodic solutions of LODE with CC using finite Fourier transforms see Kaplan [7].
3. Remarks on the resonance cases. A special attention should be paid to the so-called "resonance" case, i.e. when $P(s)$ is a divisor of zero. It is the case, when some of the roots of the polynomial $P(\lambda)$ are of the form $\frac{k \pi 1}{T}$ with $k \in \mathbb{Z}$. Then, from (10) it is seen that the solution, when it exists, is not unique. In order such a solution to exist it is necessary the condition

$$
[f+Q(s)]\left\{e^{\frac{k \pi_{1}}{T}}\right\}=0, k= \pm 1, \pm 2, \ldots
$$

to be satisfied for all the resonance roots $\frac{k \pi 1}{T}$ of $P(\lambda)$. It is equivalent to the requirements

$$
\frac{1}{T} \int_{0}^{T}\left(1-e^{\frac{k \pi_{1}}{T}}\right) f(t) d t=0, k= \pm 1, \pm 2, \ldots
$$

All these conditions are sufficient for a solution of the problem (9) to exist in the corresponding resonance case.
4. The use of the CAS Mathematica. Capabilities of the CAS Mathematica allow the Heaviside algorithm and the modified Heaviside algorithm to be implemented in a convenient way. A collection of functions is defined to provide computation and visualization of the periodic solutions of LODE with CC and of systems of such equations.

Acknowledgement. The author expresses her gratitude to Prof. I. Dimovski for proposing the problem and for the useful discussions.

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# НАМИРАНЕ НА ПЕРИОДИЧНИ РЕШЕНИЯ НА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ЧРЕЗ ПОДХОДА НА ОПЕРАЦИОННОТО СМЯТАНЕ 

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Операционното смятане на Хевисайд-Микусински може да се използува за решаване на задачи с начални условия за някои типове диференциални уравнения, чрез трансформирането им в алгебрични уравнения. Разгледано е едно разширение на алгоритъма на Хевисайд-Микусински, с цел неговото прилагане за намиране на периодични решения на линейни диференциални уравнения с постоянни коефициенти и на системи от такива уравнения. С използване на системата за компютърна алгебра Mathematica решенията се представят в символен вид.


[^0]:    *Partially supported by Grant No I-1002 of the NSF, Bulgaria.

