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ON AN INTERGRID TRANSFER OPERATOR BETWEEN NONNESTED ISOPARAMETRIC SPACES

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A second order elliptic problem on a curved domain is considered. An isoparametric approach is used for obtaining a sequence of regular triangulations. A bijection from one approximating domain to another is found. On this basis a new intergrid transfer operator is constructed. The stability and the saturation property of the considered operator are proved. The application of the new operator for obtaining of a nonnested multigrid method is discussed.

Introduction. Multigrid methods are among the most efficient methods for solving elliptic partial differential equations. In the most papers multigrid methods, which use a sequence of nested finite element spaces are considered. However, there are many problems where one has to deal with nonnested spaces, e.g., in the cases of some mixed finite element methods, some C^1 finite elements, non-quasi-uniform or degenerate triangulations, noninherited bilinear forms, curved boundaries etc. Many authors consider nonnested multigrid methods for various elliptic problems [1–3, 8–15, etc.]

The determining of the intergrid transfer operator is a basic step for constructing of nonnested multigrid methods. The intergrid transfer operator should have two important properties to be applied for compiling of a multigrid algorithm, namely stability and saturation property. These properties are thoroughly proved in the present paper.

Most of the known results are obtained by the piecewise linear elements, i.e. lowest rate of convergence is obtained. Even those of the authors who consider problems on curved domains have not used up to now isoparametric approach for finding a multigrid solution (see for example [2, 10, 11]). However, there exist many problems, where the usage of linear elements leads to divergence of the approximate solutions (see, e.g., [4, Chapter 8.2]). Therefore the application of the isoparametric approach for constructing of multigrid methods is well motivated.

Let Ω be a curved bounded domain in $\mathbf{R}^2$ with a Lipschitz-continuous boundary Γ. As usual, we denote the real Sobolev space for n nonnegative integer by $H^n(\Omega)$. The space $H^n(\Omega)$ is provided with the norm $\|\cdot\|_{n,\Omega}$ and the seminorm $|\cdot|_{n,\Omega}$. Let us define the space $H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma\}$, the energy scalar product $a(u,v) = \int_\Omega a(x)\nabla u \cdot \nabla v + b(x)uv\,dx$ and the L^2 inner product $(u,v) = \int_\Omega uv\,dx,\ u,v \in H_0^1(\Omega)$. Assume that $a \in C^1(\overline{\Omega})$, $b,f \in C(\overline{\Omega})$ and there exist positive constants $\underline{a}$, $\overline{a}$, $\underline{b}$, $\overline{b}$, such that $\underline{a} \le a(x) \le \overline{a}$, $\underline{b} \le b(x) \le \overline{b}$, $\forall x \in \overline{\Omega}$.

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Consider the following model problem:

$$(\mathcal{P}) \quad : \quad \begin{cases} \text{Find a function } u \in \mathbf{V} \text{ that satisfies} \\ a(u,v) \quad = \quad (f,v), \qquad \forall v \in \mathbf{V} \end{cases} .$$

Isoparametric finite element discretizations. Let τ_0 be an initial triangulation of the domain Ω with finite elements of degree $m \geq 2$. We assume that all finite elements in the triangulation τ_0 of the domain Ω are isoparametric equivalent to one finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ called finite element of reference with $\hat{K} = \{(\hat{x}_1, \hat{x}_2) \mid \hat{x}_1 \geq 0, \hat{x}_2 \geq 0, \hat{x}_1 + \hat{x}_2 \leq 1\}$ is the canonical 2-simplex; $\hat{P} = P_m(\hat{K})$, where P_m is the space of all polynomials of degree not exceeding m; $\hat{\Sigma} = \left\{ \hat{a} = (\hat{a}_1, \hat{a}_2) \mid \hat{a}_1 = \dfrac{i}{m},\ \hat{a}_2 = \dfrac{j}{m};\ i + j \leq m;\ i, j \in \mathbf{N} \cup \{0\} \right\}$ is the set of all Lagrangian interpolation nodes.

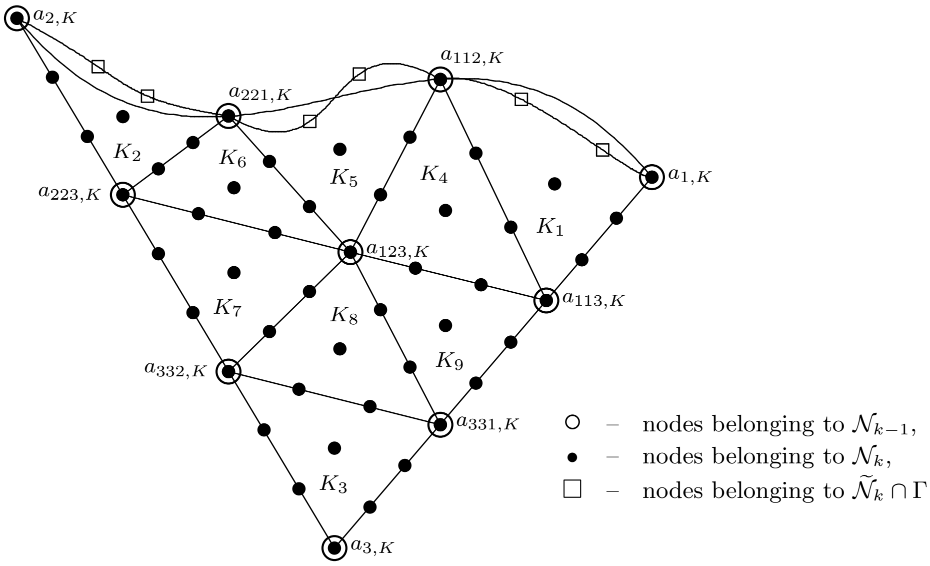

Fig. 1. The finite element $K \in \tau_{k-1}$ and the corresponding $\bigcup_{i=1}^{m^2} K_i,\ K_i \in \tau_k,\ m = 3$

An arbitrary finite element $K \in \tau_0$ is defined by $K = F_K(\hat{K})$, where $F_K \in \hat{P}^2$ is an invertible transformation. We use not only straight elements but also isoparametric elements with one curved side for getting a good approximation of the boundary Γ. Thus we obtain a domain $\Omega_0 = \bigcup_{K \in \tau_0} K$, which approximates the domain Ω.

Let $\mathcal{N}_k$, $k \in \mathbf{N}$, be the set of all nodes of the triangulation τ_k and $\widetilde{\mathcal{N}}_k = \mathcal{N}_k \setminus \mathcal{N}_{k-1}$. Let Ω_k be an approximate domain of the domain Ω corresponding to the triangulation τ_k and $\Gamma_k = \partial\Omega_k$. The triangulation τ_k is obtained from τ_{k-1} by dividing each element into m^2 elements of degree m (see Fig. 1). We can see that the nodes in $\widetilde{\mathcal{N}}_k \cap \Gamma$ do not belong to Γ_{k-1} (see Fig. 1).

We define the unique affine map $\widetilde{F}_K$ satisfying the conditions $\widetilde{F}_K(\hat{a}_i) = a_{i,K},\ \ K \in \tau_k$, where $a_{i,K}$, $i = 1, 2, 3$, are the vertex nodes of the finite element K. Denote $\widetilde{K} = \widetilde{F}_K(\hat{K})\,,\ \ h_K = \operatorname{diam}(\widetilde{K})\,,\ \ \forall K \in \tau_k,\ \ h_k = \max_{K \in \tau_k} h_K$.

Suppose that he following hypotheses are valid.

H1 The boundary Γ is piecewise C^{m+1}.

H2 The triangulation τ_k consists of isoparametric elements of degree $m \geq 2$.

H3 The triangulation τ_k is m-regular in the sense of Ciarlet and Raviart [6].

H4 The sets of the nodes of two consecutive triangulations are connected by $\mathcal{N}_{k-1} \subset \mathcal{N}_k$ (see Fig. 1).

H5 All the nodes $a_i \in \mathcal{N}_k \cap \Gamma_k$ belong to Γ too. Usually the nodes $a_i \in \widetilde{\mathcal{N}}_k \bigcap \Gamma_k$ do not belong to Γ_{k-1}.

Define a finite element space $\mathbf{V}_k$ associated with a triangulation τ_k by $\mathbf{V}_k = \{v \in C(\Omega_k) \mid v(x) = 0,\ x \in \Gamma_k;\quad v_{|K} \in P_K,\ K \in \tau_k\}$, where $P_K = \{p : K \to \mathbf{R} \mid p = \hat{p} \circ F_K^{-1},\ \hat{p} \in \hat{P}\}$.

Since the domain Ω is bounded there exists an open set $\widetilde{\Omega}$, which satisfies $\Omega \subset \widetilde{\Omega},\ \Omega_k \subset \widetilde{\Omega}$ for all considered triangulations τ_k. We define the function $\widetilde{a}(x)$ as a smooth extension of the coefficient $a(x)$ and functions $\widetilde{b}(x),\ \widetilde{f}(x)$ as continuous extensions of $f(x),\ b(x)$ on $\widetilde{\Omega}$. Then, we introduce the approximating bilinear form by

$$a_k(u,v) = \int_{\Omega_k} \widetilde{a}(x)\nabla u \cdot \nabla v + \widetilde{b}(x)uv\,dx\,,\quad \forall u,v \in \mathbf{H}_0^1(\Omega_k).$$

The L^2-scalar product in $\mathbf{V}_k$ is defined by $(u,v)_k = \int_{\Omega_k} uv\,dx\,,\quad \forall u,v \in \mathbf{V}_k$. Suppose that the following hypothesis is valid.

H6 The bilinear forms $a_k(\cdot,\cdot)$ are uniformly $\mathbf{V}_k$-elliptic.

Prolongations and restrictions. In this section we define an intergrid transfer operator I_k from $\mathbf{V}_{k-1}$ to $\mathbf{V}_k$. The operator I_k plays an essential role in the nonnested multigrid algorithm. For that reason we investigate the properties of I_k.

We introduce the map $\chi_k : \Omega_{k-1} \longrightarrow \Omega_k$ by $\chi_k = \Phi_k^{-1} \circ \Phi_{k-1}$, where $\Phi_k : \Omega_k \longrightarrow \Omega$ is the map defined by Lenoir [7]. Define the intergrid operator $I_k : \mathbf{V}_{k-1} \longmapsto \mathbf{V}_k$ by $I_k v = v \circ \chi_k^{-1}$.

Consider the eigenpairs $(\lambda_i^{(k)}, \psi_i^{(k)})$ of the problem

$$a_k(\psi_i^{(k)}, v) = \lambda_i^{(k)} (\psi_i^{(k)}, v)_k\,,\quad \forall v \in \mathbf{V}_k,$$

$i = 1, 2, \dots,\ N_k = \dim \mathbf{V}_k$. Without loss of generality we suppose that the eigenfunctions are normalized by the following way

$$(\psi_i^{(k)}, \psi_j^{(k)})_k = \delta_{ij}\,,\quad a_k(\psi_i^{(k)}, \psi_j^{(k)}) = \lambda_i^{(k)}\delta_{ij},$$

where δ_{ij} is the Kronecker symbol. For each v in $\mathbf{V}_k$ we have the following representation $v = \sum_{i=1}^{N_k} c_i \psi_i^{(k)}$. Define the norm $|||\cdot|||_{s,k}$ by $|||v|||_{s,k}^2 = \sum_{i=1}^{N_k} c_i^2 \Big(\lambda_i^{(k)}\Big)^s, \quad s \geq 0$. Obviously, in $\mathbf{V}_k$ we have

$$|||\cdot|||_{0,k} = ||\cdot||_{0,\Omega_k} \ \text{ and } \ |||\cdot|||_{1,k} = |||\cdot|||_k \simeq ||\cdot||_{1,\Omega_k}, \tag{1}$$

where $|||\cdot|||_k$ is the usual energy norm.

Further we shall use the classical denotations for the seminorms $|F|_{n,\infty,K}$ of a n-times differentiable map F [4, Section 4.3] and shall drop the index $\mathcal{L}_n(\mathbf{R}^2;\mathbf{R}^2)$ of the norms of the Fréchet derivatives. In other words we shall write only $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{L}_n(\mathbf{R}^2;\mathbf{R}^2)}$.

Theorem 1. *Stability. Let the hypotheses* **H1**–**H6** *be valid, then*

$$|||I_k v|||_{s,k} \lesssim |||v|||_{s,k-1}, \quad \forall v \in \mathbf{V}_{k-1}\,,\ k \in \mathbf{N}\,,\ s = 0,1.$$

Proof. We estimate $I_k v$ locally in terms of Sobolev norms

$$||I_k v||^2_{0,\chi_K(K)} = \int_{\chi_K(K)} (I_k v)^2\, dy = \int_K v^2\, |J_{\chi_k}(x)|\; dx \lesssim |J_{\chi_k}|_{0,\infty,K} \|v\|^2_{0,K},$$

$\forall K \in \tau_{k-1}$ and $y = \chi_k(x)$. Let $\mathcal{K} = \Phi_{k-1}(K)$. Since H2 holds, we have (see e.g. [7])

$$|J_{\chi_k}|_{0,\infty,K} \lesssim \left| J_{\Phi_K^{-1}} \right|_{0,\infty,\mathcal{K}} |J_{\Phi_{k-1}}|_{0,\infty,K} = O(1), \tag{2}$$

$$\left|\chi_k^{-1}\right|_{0,\infty,\chi_k(K)} \lesssim |\Phi_k|_{0,\infty,\chi_k(K)} \left|\Phi_{k-1}^{-1}\right|_{0,\infty,\mathcal{K}} = O(1). \tag{3}$$

Then $||I_k v||_{0,\infty,\Omega_k} \lesssim ||v||_{0,\infty,\Omega_{k-1}}$.

We continue with the local estimate of the first Sobolev seminorm of $I_k v$ on $\chi_k(K)$

$$\begin{aligned} |I_k v|^2_{1,\chi_K(K)} &= \int_{\chi_K(K)} (\nabla I_k v)^2\, dy \lesssim \int_K \left|\left|D(v \circ \chi_k^{-1})\right|\right|^2 |J_{\chi_k}(x)|\; dx \\ &\lesssim |J_{\chi_k}|_{0,\infty,K} \left|\chi_k^{-1}\right|^2_{0,\infty,\chi_k(K)} \int_K ||Dv||^2\, dx \;\lesssim\; |v|^2_{1,K}. \end{aligned}$$

Thus we obtain $||I_k v||_{s,\chi_K(K)} \lesssim ||v||_{s,K}$, $s = 0,1$. Summing over all elements $K \in \tau_{k-1}$ we have for $s = 0,1$

$||I_k v||_{s,\Omega_k} = \left(\sum_{K\in\tau_{k-1}} ||I_k v||^2_{s,\chi_K(K)}\right)^{1/2} \lesssim \left(\sum_{K\in\tau_{k-1}} ||v||^2_{s,K}\right)^{1/2} = ||v||_{s,k-1}.$

Using the norm equivalence (1) we complete the proof. $\square$

Theorem 2. *Saturation property. Let the triangulations* τ_k *satisfy the conditions* **H1**–**H6**. *Then the inequality*

$$|||w - I_k v|||_k \lesssim |||w \circ \chi_k - v|||_{k-1}\,, \quad \forall v \in \mathbf{V}_{k-1}\,, \quad \forall w \in \mathbf{V}_k \tag{4}$$

holds.

Proof. Replacing w by $z + I_k v$, $z \in \mathbf{V}_k$, we transform (4) into

$$|||z|||_k \lesssim |||z \circ \chi_k|||_{k-1}\,, \quad \forall z \in \mathbf{V}_k\,. \tag{5}$$

For verifying the inequality (4), it is enough to prove (5). We use the same approach as in the proof of Theorem 1. Using again (2) and (3), we obtain

$$\begin{aligned} |z|^2_{1,\chi_k(K)} &= \int_{\chi_k(K)} (\nabla z)^2\, dy \;\lesssim\; \int_K \left|\left|D(z\circ\chi_k) D\chi_k^{-1}\right|\right|^2 |J_{\chi_k}(x)|\; dx \\ &\lesssim |J_{\chi_k}|_{0,\infty,K} \left|\chi_k^{-1}\right|^2_{0,\infty,\chi_k(K)} \int_K ||D(z\circ\chi_k)||^2\, dx \;\lesssim\; |z\circ\chi_k|^2_{1,K}\,. \end{aligned}$$

Summing over all elements $K \in \tau_{k-1}$, we have $|z|_{1,\Omega_k} \lesssim |z \circ \chi_k|_{1,\Omega_{k-1}}$.

Applying the norm equivalence (1) we get the validity of (5). $\square$

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ВЪРХУ ЕДИН МЕЖДУМРЕЖОВ ТРАНСФЕРЕН ОПЕРАТОР ОПРЕДЕЛЕН ЧРЕЗ НЕВЛОЖЕНИ ИЗОПАРАМЕТРИЧНИ ПРОСТРАНСТВА

Тодор Д. Тодоров

Разгледна е моделна задача за елиптичен оператор от втори ред върху криволинейна област с непрекъсната по Липшиц граница. За получаване на редица от регулярни триангулации е използван изопараметричен подход. Установена е биекция между две последователни трианглуции в тази редица. На тази база е въведен нов междумрежов трансферен оператор. За този оператор са доказани свойствата устойчивост и сатурация, които са необходими за конструиране на невложен многомрежов метод.