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# ON THE EQUIAFFINE CONNECTIONS<sup>\*</sup>

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In this paper, an average equiaffine connection is defined as the average of n symmetric connections and second equiaffine connection is defined, with Ricci tensor, for any equiaffine connection.

Using the Ricci tensor, a second equiaffine connection is specified for any equiaffine connection. The projective transformation of these equiaffine connections is studied. Next, it is determined the category of the spaces EqAn and  ${}^{1}EqAn$ , the connections of which are in projective equivalence, in the case when they are spaces of a special composition.

**1. Preliminaries.** A space with an affine connection without torsion is equiaffine, provided that the parallel transformation of n vectors  $\overset{i}{\overset{v}{\alpha}}(\alpha = 1, 2, ..., n)$  preserves the volume, where the volume is [1,p.150]:

(1) 
$$V = e_{i_1 i_2 \dots i_n} \frac{v^{i_1} v^{i_2} \dots v^{i_n}}{1 2} \dots \frac{v^{i_n}}{n}.$$

Here,  $e_{i_1i_2...i_n}$  is called fundamental *n*-vector of the space.

Suppose that EqAn is an equiaffine space with a fundamental *n*-vector, coefficients of a connection  $\Gamma_{is}^k$ , a tensor of curvature  $R_{isk}^m$ , and a tensor of Ricci  $R_{is}$ . We denote the fundamental form of the fundamental *n*-vector by  $e = e_{12...n}$ . The equiaffine space EqAnis characterized by one of the following equalities [1, p. 166]

(2) 
$$\nabla_k e_{i_1 i_2 \dots i_n} = 0$$

(3) 
$$\Gamma_{ks}^s = \partial_k \ln e$$

(4) 
$$R_{is} = R_{si},$$

(5) 
$$R_{isk}^k = 0.$$

Let there be given n symmetric connections  ${}^{\alpha}\Gamma_{is}^{k}(\alpha = 1, 2, ..., n)$  and an arbitrary tensor  $A_{i_1i_2...i_n}$  in the differentiable manifold  $X_n$ . Then

(6) 
$${}^{\alpha}\nabla_{k}A_{i_{1}i_{2}...i_{n}} = \partial_{k}A_{i_{1}i_{2}...i_{n}} - {}^{\alpha}\Gamma_{ki_{1}}^{m}A_{mi_{2}...i_{n}} - {}^{\alpha}\Gamma_{ki_{2}}^{m}A_{i_{1}mi_{3}...i_{n}} - \cdots - {}^{\alpha}\Gamma_{ki_{n}}^{m}A_{i_{1}i_{2}...i_{n-1}m}.$$

The mixed covariant derivative is [1, p. 136]:

(7) 
$$\nabla_k A_{i_1 i_2 \dots i_n} = \partial_k A_{i_1 i_2 \dots i_n} - {}^1\Gamma^m_{k i_1} A_{m i_2 \dots i_n} - {}^2\Gamma^m_{k i_2} A_{i_1 m i_3 \dots i_n} - \cdots - {}^n\Gamma^m_{k i_n} A_{i_1 i_2 \dots i_{n-1} m}.$$

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Equiaffine connections are treated in [4] and [5]. In [5], an apparatus is developed to study the geometry of compositions in equiaffine spaces.

Let EqAn be an equiaffine space of compositions  $X_k \times X_{n-k}$ , defined with an affinor of the composition  $a_{\alpha}^{\beta}$  [2]. The special compositions in EqAn are studied by means of the tensor  $\overline{R} = a_{\alpha}^{\sigma} R_{\sigma}^{\beta}$  [5]. Now, we denote with  $P(X_k)$  and  $P(X_{n-k})$  the position of the fundamental manifolds  $X_k$  and  $X_{n-k}$  of the composition  $X_k \times X_{n-k}$ . Then we define the composition  $X_k \times X_{n-k}$  in EqAn as an R-composition, if for arbitrary vectors  $v^{\alpha} \in P(X_k)$ and  $v^{\alpha} \in P(X_{n-k})$  the equation  $R_{\alpha\beta} v^{\alpha} v^{\beta} = 0$  holds [5]. It is proved that R-compositions in EqAn satisfy the condition:

(8) 
$$\bar{R}_{\alpha\beta} = \bar{R}_{\beta}$$

**2.** Average equiaffine connection. Suppose that there are *n* symmetric connections  ${}^{\alpha}\Gamma_{is}^{k}(\alpha = 1, 2, ..., n)$  and an *n*-vector  $e_{i_{1}i_{2}...i_{n}}$  in the differentiable manifold Xn.

**Theorem 1.** If the volume (1) is preserved by the parallel transfer of arbitrary vectors  $\overset{i}{v}, \overset{i}{v}, \ldots, \overset{i}{v}$  by the connections  ${}^{1}\Gamma_{is}^{k}, {}^{2}\Gamma_{is}^{k}, \ldots, {}^{n}\Gamma_{is}^{k}$  and  ${}^{2}\Gamma_{is}^{k}, {}^{3}\Gamma_{is}^{k}, \ldots, {}^{n}\Gamma_{is}^{k}, {}^{1}\Gamma_{is}^{k}$  and  ${}^{n}\Gamma_{is}^{k}, {}^{1}\Gamma_{is}^{k}$  and  ${}^{n}\Gamma_{is}^{k}, {}^{1}\Gamma_{is}^{k}$  and  ${}^{n}\Gamma_{is}^{k}, \ldots, {}^{n-1}\Gamma_{is}^{k}$ , then the average connection  $\Gamma_{is}^{k} = \frac{1}{n}({}^{1}\Gamma_{is}^{k} + {}^{2}\Gamma_{is}^{k} + \cdots + {}^{n}\Gamma_{is}^{k})$  of the connections  ${}^{1}\Gamma_{is}^{k}, {}^{2}\Gamma_{is}^{k}, \ldots, {}^{n}\Gamma_{is}^{k}$  is equiaffine with a fundamental n-vector  $e_{i_{1}i_{2}...i_{n}}$ .

**Proof.** Under the hypothesis of Theorem 1, from dV = 0, when the vectors  $v_1^i, v_2^i, \dots, v_n^i$  are transfered parallelly in the respective connections, it follows that  $\delta e_{i_1 i_2 \dots i_n} = 0$ . Since the last equality should be fulfilled independently of the choice of  $du^k$ , it is equivalent to

$$(9) \begin{array}{c} \nabla_{k}e_{i_{1}i_{2}\ldots i_{n}} = \partial_{k}e_{i_{1}i_{2}\ldots i_{n}}^{-1}\Gamma_{ki_{1}}^{m}e_{mi_{2}\ldots i_{n}}^{-2}\Gamma_{ki_{2}}^{m}e_{i_{1}mi_{3}\ldots i_{n}}^{-\cdots -n}\Gamma_{ki_{n}}^{m}e_{i_{1}i_{2}\ldots i_{n-1}m} \\ \nabla_{k}e_{i_{1}i_{2}\ldots i_{n}} = \partial_{k}e_{i_{1}i_{2}\ldots i_{n}}^{-2}\Gamma_{ki_{1}}^{m}e_{mi_{2}\ldots i_{n}}^{-3}\Gamma_{ki_{2}}^{m}e_{i_{1}mi_{3}\ldots i_{n}}^{-\cdots -1}\Gamma_{ki_{n}}^{m}e_{i_{1}i_{2}\ldots i_{n-1}m} \\ \cdots \\ \nabla_{k}e_{i_{1}i_{2}\ldots i_{n}} = \partial_{k}e_{i_{1}i_{2}\ldots i_{n}}^{-n}\Gamma_{ki_{1}}^{m}e_{mi_{2}\ldots i_{n}}^{-1}\Gamma_{ki_{2}}^{m}e_{i_{1}mi_{3}\ldots i_{n}}^{-\cdots -1}\Gamma_{ki_{n}}^{m}e_{i_{1}i_{2}\ldots i_{n-1}m} \end{array}$$

If at least two of the indices  $i_p$  and  $i_s$  are equal in some equation, then the system is satisfied identically. Therefore, it remains to consider the equations in (9), which correspond to different indices. These because of the asymmetry of the *n*-vector  $e_{i_1i_2...i_n}$ , are equivalent to:

(10) 
$$\begin{array}{l} \partial_{k}e_{12...n} - {}^{1}\Gamma_{k1}^{m}e_{m2...n} - {}^{2}\Gamma_{k2}^{m}e_{1m3...n} - \cdots - {}^{n}\Gamma_{kn}^{m}e_{12...n-1m} = 0\\ \partial_{k}e_{12...n} - {}^{2}\Gamma_{k1}^{m}e_{m2...n} - {}^{3}\Gamma_{k2}^{m}e_{1m3...n} - \cdots - {}^{1}\Gamma_{kn}^{m}e_{12...n-1m} = 0\\ \dots\\ \partial_{k}e_{12...n} - {}^{n}\Gamma_{k1}^{m}e_{m2...n} - {}^{1}\Gamma_{k2}^{m}e_{1m3...n} - \cdots - {}^{n-1}\Gamma_{kn}^{m}e_{12...n-1m} = 0. \end{array}$$

By substituting the fundamental form  $e = e_{12...n}$  and discarding the zero terms, we obtain:

(11) 
$${}^{1}\Gamma_{k1}^{1} + {}^{2}\Gamma_{k2}^{2} + \dots + {}^{n-1}\Gamma_{kn-1}^{n-1} + {}^{n}\Gamma_{kn}^{n} = \partial_{k}\ln e \\ {}^{2}\Gamma_{k1}^{1} + {}^{3}\Gamma_{k2}^{2} + \dots + {}^{n}\Gamma_{kn-1}^{n-1} + {}^{1}\Gamma_{kn}^{n} = \partial_{k}\ln e \\ \dots \\ {}^{n}\Gamma_{k1}^{1} + {}^{1}\Gamma_{k2}^{2} + \dots + {}^{n-2}\Gamma_{kn-1}^{n-1} + {}^{n-1}\Gamma_{kn}^{n} = \partial_{k}\ln e.$$

The conditions (11) are necessary and sufficient to preserve the volume (1) by the 142

parallel transfer of arbitrary vectors  $v_1^i, v_2^i, \ldots, v_n^i$ , in the connections  ${}^1\Gamma_{is}^k, \ldots, {}^n\Gamma_{is}^k$ ,  ${}^2\Gamma_{is}^k, \ldots, {}^n\Gamma_{is}^k, {}^1\Gamma_{is}^k, \operatorname{and} {}^n\Gamma_{is}^k, {}^1\Gamma_{is}^k, \ldots, {}^{n-1}\Gamma_{is}^k$ , respectively. Finally by adding (11) term-wise, we obtain:  ${}^1\Gamma_{ks}^s + {}^2\Gamma_{ks}^s + \cdots + {}^{n-1}\Gamma_{ks}^s + {}^n\Gamma_{ks}^s = n\partial_k \ln e$ , i.e.

(12) 
$$\frac{1}{n} \left( {}^{1}\Gamma^{s}_{is} + {}^{2}\Gamma^{s}_{is} + \dots + {}^{n}\Gamma^{k}_{is} \right) = \partial_{k} \ln e$$

From (12) it follows that the average connection  $\frac{1}{n} \left( {}^{1}\Gamma_{is}^{s} + {}^{2}\Gamma_{is}^{s} + \dots + {}^{n}\Gamma_{is}^{k} \right)$  of the connections  ${}^{1}\Gamma_{is}^{k}$ ,  ${}^{2}\Gamma_{is}^{k}$ ,  $\dots$ ,  ${}^{n}\Gamma_{is}^{k}$  is equiaffine with fundamental *n*-vector  $e_{i_{1}i_{2}...i_{n}}$ .

**3. Transformation of equiaffine connections.** Let there be given an equiaffine space EqAn with connection  $\Gamma_{is}^k$ , a fundamental *n*-vector  $e_{i_1i_2...i_n}$ , a tensor of curvature  $R_{isk}^i$ , and a Ricci tensor  $R_{is}$ . Consider the following connection

(13) 
$${}^{1}\Gamma^{k}_{is} = \Gamma^{k}_{is} + R_{is}R^{kj}\partial_{j}\ln e$$

where  $R^{is}$  is the reciprocal tensor to  $R_{is}$  and  $e = e_{12...n}$ . The symmetry of the Ricci tensor implies that the connection  ${}^{1}\Gamma_{is}^{k}$  is symmetric.

**Theorem 2.** The connection  ${}^{1}\Gamma_{is}^{k}$  is equiaffine with a fundamental form  $e^{2}$ .

**Proof.** From (13) it follows that  ${}^{1}\Gamma_{is}^{k} = \Gamma_{is}^{k} + R_{is}R^{kj}\partial_{j}\ln e$ , and taking into account (3), we obtain

(14) 
$${}^{1}\Gamma^{k}_{ik} = \partial_i \ln e^2.$$

So that the connection  ${}^{1}\Gamma_{is}^{k}$  is equiaffine with a fundamental form  $e^{2}$ .

If we denote by  ${}^{1}R_{isk.}^{j}$  and  ${}^{1}R_{sk}$  the tensor of curvature and the tensor of Ricci, respectively, for the connection  ${}^{1}\Gamma_{is}^{k}$ , then some straightforward arithmetic gives

(15) 
$${}^{1}R_{isk.}^{j} = R_{isk.}^{j} + 2\nabla_{[i}(R_{s]k}R^{jp}\partial_{p}\ln e) + 2R^{jp}R_{k[s}\partial_{i]}\ln e\partial_{p}\ln e,$$
  
$${}^{1}R_{sk} = R_{sk} + \nabla_{i}(R_{sk}R^{ip}\partial_{p}\ln e) + R^{ip}R_{sk}\partial_{i}\ln e\partial_{p}\ln e.$$

From the first equation of (15), after contracting on the indices k and j, we obtain that  $R_{isk.}^k = {}^1R_{isk.}^k$ , which was to be expected since for the equiaffine connections  $\Gamma_{is}^k$  and  ${}^1\Gamma_{is}^k$  it holds that  $R_{isk.}^k = 0$  and  ${}^1R_{isk.}^k = 0$ .

**Theorem 3.** If there exists a projective transformation between the equiaffine connections  $\Gamma_{is}^k$  and  ${}^1\Gamma_{is}^k$ , then

(16) 
$${}^{1}R_{isk.}^{j} = R_{isk.}^{j} + \frac{2}{n+1}\nabla\delta_{[s}^{j}\nabla_{i]}\partial_{k}\ln e + \frac{2}{n+1}\nabla\delta_{[s}^{j}\partial_{i]}\ln e)\partial_{k}\ln e.$$

**Proof.** Let there be a projective correspondence between the equiaffine connections  $\Gamma_{is}^k$  and  ${}^{1}\Gamma_{is}^k$ . According to [1, p. 166] there is projective transformation between the connections  $\Gamma_{ik}^k$  and  ${}^{1}\Gamma_{ik}^k$  if and only if the tensor of affine deformation  $T_{is}^k = {}^{1}\Gamma_{is}^k - \Gamma_{is}^k$  satisfies

(17) 
$$T_{is}^{k} = (\delta_{i}^{k} \delta_{s}^{j} + \delta_{s}^{k} \delta_{i}^{j}) P_{j}$$

where

(18) 
$$p_j = \frac{1}{n+1} T_{jm}^m.$$

From (13), (17) and (18) follows

(19) 
$$T_{is}^{k} = R_{is}R^{kj}\partial_{j}\ln e = \frac{n-1}{n+1}(\delta_{i}^{k}\delta_{s}^{j} + \delta_{s}^{k}\delta_{i}^{j})\partial_{j}\ln e.$$

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Substituting  $R_{is}R^{kj}\partial_i \ln e$  from (19) into the first equality of (15) and taking into account that  $\nabla m(\delta_i^k \delta_s^j + \delta_s^k \delta_{is}^j) = 0$ , we obtain (16). From (16) we find that

(20) 
$${}^{1}R_{sk} = R_{sk} - \frac{n-1}{n+1}\partial_s\partial_k \ln e - \frac{n-1}{n+1}\ln e\partial_k \ln e.$$

We denote by  ${}^{1}EqAn$  the equiaffine connection defined by the connection  ${}^{1}\Gamma_{is}^{k}$ .

The fundamental form of  ${}^{1}EqAn$  is  $e^{2}$ . The spaces EqAn and  ${}^{1}EqAn$  are differential manifolds in Xn equipped with the equiaffine connections  $\Gamma_{is}^{k}$  and  ${}^{1}\Gamma_{is}^{k}$ , respectively.

Suppose that in Xn there is an affinor  $a^{\beta}_{\alpha}$  which satisfies the condition

(21) 
$$a^{\sigma}_{\alpha}a^{\beta}_{\sigma} = \delta^{\beta}_{\alpha}.$$

According to [3], in the equiaffine spaces EqAn and  ${}^{1}EqAn$ , the affinor  $a_{\alpha}^{\beta}$  defines compositions  $X_{k} \times X_{n-k}$  of two fundamental manifolds  $X_{k}$  and  $X_{n-k}$ .

The *R*-compositions in EqAn and  ${}^{1}EqAn$  are characterized by:

(22) 
$$\bar{R}_{[\alpha\beta]} = 0, {}^1R_{[\alpha\beta]} = 0,$$

where

(23) 
$$\bar{R}_{[\alpha\beta]} = a^{\sigma}_{\alpha} R_{\sigma\beta}, {}^{1}\bar{R}_{[\alpha\beta]} = a^{\sigma}_{\alpha} {}^{1}R_{\sigma\beta}.$$

In [3] is introduced an adapted coordinate system with the composition  $X_k \times X_{n-k}(u^i, u^{\overline{i}})$ ,  $(i = 1, 2, ..., k; \overline{i} = k + 1, ..., n)$ . The matrix of the affinor  $a_{\alpha}^{\beta}$  of the composition in the adapted coordinate system has the form [3]:

(24) 
$$a_{\alpha}^{\beta} = \begin{pmatrix} \delta_{i}^{j} & 0\\ 0 & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}.$$

After multiplying both sides of (20) with the affinor  $a^{\beta}_{\alpha}$ , and taking into account (23), we obtain

(25) 
$${}^{1}\bar{R}_{\alpha\beta} = \bar{R}_{\alpha\beta} - \frac{n-1}{n+1}a^{\sigma}_{\alpha}(\partial_{\sigma}\partial_{\beta}\ln e + \partial_{\sigma}\ln e\partial_{\beta}\ln e).$$

**Theorem 4.** If the composition  $X_k \times X_{n-k}$  is an *R*-composition in the spaces EqAn and <sup>1</sup>EqAn, which have a projective transformation, then in the adapted coordinate system the fundamental form of the *n*-vector  $e_{i_1i_2...i_n}$  is defined as follows:

(26) 
$$e = f(\frac{1}{u}) + f(\frac{2}{u}) + \dots + f(\frac{n}{u}),$$

where  $f(\overset{1}{u})$  are arbitrary functions.

**Proof.** Since it is given that the spaces EqAn and  ${}^{1}EqAn$  are in projective transformation, (25) holds. By (22) and (25) the composition  $X_k \times X_{n-k}$  is an R-composition if and only if the tensor

(27) 
$$b_{\alpha\beta} = a^{\sigma}_{\alpha} (\partial_{\sigma} \partial_{\beta} \ln e + \partial_{\sigma} \ln e \partial_{\beta} \ln e)$$

is symmetric. From (24) and (27) follows that in the adapted coordinate system we have:

Hence,  $b_{[\alpha\beta]} = 0$  if and only if

(29) 
$$\partial_i \partial_{-k} \ln e + \partial_i \ln e \partial_{-k} \ln e = 0.$$

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Solving the system of differential equation (29), we arrive at (26). Now, by (3) and (14), the equality (26) requires a restriction of the connections  $\Gamma_{is}^k$  and  ${}^{1}\Gamma_{is}^k$ , i.e. it specializes the spaces EqAn and  ${}^{1}EqAn$ .

#### REFERENCES

 A. P. NORDEN. Affinely Connected Spaces, GRFML, Moscow, 1976 (in Russian).
A. P. NORDEN. Spaces of Cartesian compositions, *Izv. Vyssh. Uchebn. Zaved. Mathem.* N<sup>a</sup> 4, 1963, 117-118 (in Russian).
A. P. NORDEN, G. TIMOFEEV. The invariants criterion for specials compositions in manydimensional spaces, *Izv. Vyssh. Uchebn. Zaved. Mathem.*, N<sup>a</sup>8, 1972, 81-89 (in Russian).
G. ZLATANOV, B. TSAREVA. Geometry of the Nets in Equiaffine Space, *Journal of Geometry*, 55 (1996), 192–201.
G. ZLATANOV G. Equiaffine Spaces of compositions, *Tensor N.S.*, 62 (2000), 189–190.

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# ВЪРХУ ЕКВИАФИННИТЕ СВЪРЗАНОСТИ

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В работата е определена еквиафинна свързаност, която е средна свързаност на nдадени симетрични свързаности.

За всяка еквиафинна свързаност с помощта на тензора на Ричи е определена втора еквиафинна свързаност. Изучено е проективното преобразование на тези две свързаности.

Еквиафинните пространства, на които свързаностите се намират в проективно съответствие, са означени с EqAn и  ${}^{1}EqAn$ . Определен е видът на пространствата EqAn и  ${}^{1}EqAn$ , когато те са пространства от специална композиция.