МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2006
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2006
Proceedings of the Thirty Fifth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 5–8, 2006

MEASURABILITY OF SETS OF PAIRS OF PARALLEL NON-ISOTROPIC STRAIGHT LINES OF THE FIRST TYPE IN THE SIMPLY ISOTROPIC SPACE*

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We study the measurability of sets of pairs of parallel non-isotropic straight lines in different isotropic planes and the corresponding invariant densities with respect to the group of the general similitudes and some its subgroups.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is defined (see [3]) as a projective space $\mathbb{P}_3(\mathbb{R})$ with an absolute consisting a plane ω (the *absolute plane*) and two complex conjugate straight lines (the *absolute lines*) f_1, f_2 into ω. The absolute lines f_1 and f_2 intersect in a real point F (the *absolute point*). In homogeneous coordinates (x_0, x_1, x_2, x_3) we can choose the plane $x_0 = 0$ as the plane ω, the line $x_0 = 0, x_1 + ix_2 = 0$ as the line f_1 and the line $x_0 = 0, x_1 - ix_2 = 0$ as the line f_2. Then the absolute point F has homogeneous coordinates $(0, 0, 0, 1)$. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of *the general simply isotropic similitudes.* Passing on to affine coordinates (x, y, z) any similitude of G_8 can be written in the form [3; p. 3]

$$
\begin{aligned}
\overline{x} &= c_1 + c_7(x\cos\varphi - y\sin\varphi),\\
\overline{y} &= c_2 + c_7(x\sin\varphi + y\cos\varphi),\\
\overline{z} &= c_3 + c_4x + c_5y + c_6z,
\end{aligned}
\tag{1}
$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \neq 0, c_7 > 0$ and φ are real parameters.

A straight line is said to be (completely) *isotropic* if its infinite point coincides with the absolute point F; otherwise, the straight line is said to be *non-isotropic* [3, p. 5].

We will consider G_8 and the following its subgroups:

I. $B_7 \subset G_8 \iff c_7 = 1$. This is the group of the simply isotropic similitudes of the δ-distance [3, p. 5].

II. $S_7 \subset G_8 \iff c_6 = 1$. This is the group of the simply isotropic similitudes of the s-distance [3, p. 6].

III. $W_7 \subset G_8 \iff c_6 = c_7$. This is the group of the simply isotropic angular similitudes [3, p. 18].

IV. $G_7 \subset G_8 \iff \varphi = 0$. This is the group of the simply isotropic boundary similitudes [3, p. 8].

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*2000 Mathematics Subject Classification: 53C65

V. $V_7 \subset G_8 \iff c_6 c_7^2 = 1$. This is the group of the simply isotropic volume preserving similitudes [3, p. 8].

VI. $G_6 = G_7 \cap V_7$. This is the group of the simply isotropic volume preserving boundary similitudes [3, p. 8].

VII. $B_6 = B_7 \cap G_7$. This is the group of the modular boundary motions [3, p. 9].

VIII. $B_5 = B_7 \cap S_7 \cap G_7$. This is the group of the unimodular boundary motions [3, p. 9].

Two points P_1 and P_2 are called *parallel* if the straight line P_1P_2 is isotropic.

We emphasize that most of the common material of the geometry of the simply isotropic space $I_3^{(1)}$ can be found in [3].

Using some basic concepts of the integral geometry in the sense of M. I. Stoka [4] and G. I. Drinfel'd [2], we study the measurability of sets of pairs of parallel non-isotropic straight lines in different isotropic planes with respect to G_8 and indicated above subgroups.

2. Measurability with respect to G_8. Let (G_1, G_2) be a pair of parallel non-isotropic straight lines

$$G_1 : x = az + p_1, y = bz + q_1, \qquad a^2 + b^2 \neq 0, \tag{2}$$
$$G_2 : x = az + p_2, y = bz + q_2.$$

The pair (G_1, G_2) is said to be of the *first type* if G_1 and G_2 lie into different isotropic planes. Then the following inequality holds:

$$a(q_2 - q_1) - b(p_2 - p_1) \neq 0. \tag{3}$$

We can assume without loss of generality that $a \neq 0$, and in this case we can take the Plücker coordinates ([3;p.38-41]) $g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2$ as the parameters of the set of pairs (G_1, G_2), where

$$g_2^1 = \frac{b}{a},\ g_3^1 = \frac{1}{a},\ g_5^1 = -\frac{p_1}{a},\ g_6^1 = -q_1 + \frac{bp_1}{a}, \tag{4}$$
$$g_5^2 = -\frac{p_2}{a},\ g_6^2 = -q_2 + \frac{bp_2}{a}.$$

Under the action of (1) the pair $(G_1, G_2)(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$ is transformed into the pair $(\overline{G}_1, \overline{G}_2)(\overline{g}_2^1, \overline{g}_3^1, \overline{g}_5^1, \overline{g}_6^1, \overline{g}_5^2, \overline{g}_6^2)$ as follows:

$$\begin{aligned}
&\overline{g}_2^1 = Kc_7(\sin\varphi + g_2^1 \cos\varphi),\\
&\overline{g}_3^1 = K(c_4 + c_5 g_2^1 + c_6 g_3^1),\\
&\overline{g}_5^i = K\{(c_3 - c_5 g_6^i + c_6 g_5^i)c_7 \cos\varphi -\\
&\qquad -[c_3 + c_4 g_6^i + c_6(g_2^1 g_5^i + g_3^1 g_6^i)]c_7 \sin\varphi - c_1(c_4 + c_5 + c_6 g_3^1)\},\\
&\overline{g}_6^i = K[(c_1 g_2^1 - c_2)\cos\varphi + (c_1 + c_2 g_2^1)\sin\varphi + c_7 g_6^i],
\end{aligned} \tag{5}$$

where $K = [c_7(\cos\varphi - g_2^1 \sin\varphi]^{-1}$ and $i = 1, 2$. The transformations (5) form the associated group $\overline{G}_8$ of G_8 [4; p. 34]. The group $\overline{G}_8$ is isomorphic to G_8 and the invariant density with respect to G_8 of the pairs of lines (G_1, G_2), if it exists, coincides with the density with respect to $\overline{G}_8$ of the points $(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$ in the set of parameters.

The associated group $\overline{G}_8$ has the infinitesimal operators

$$
(6)\quad
\begin{aligned}
&Y_1 = g_3^1\frac{\partial}{\partial g_5^1} - g_2^1\frac{\partial}{\partial g_6^1} + g_3^1\frac{\partial}{\partial g_5^2} - g_2^1\frac{\partial}{\partial g_6^2},\ \ Y_2 = \frac{\partial}{\partial g_6^1} + \frac{\partial}{\partial g_6^2},\\
&Y_3 = \frac{\partial}{\partial g_5^1} + \frac{\partial}{\partial g_5^2},\ \ Y_4 = \frac{\partial}{\partial g_3^1},\ \ Y_5 = g_2^1\frac{\partial}{\partial g_3^1} - g_6^1\frac{\partial}{\partial g_5^1} - g_6^2\frac{\partial}{\partial g_5^2},\\
&Y_6 = g_3^1\frac{\partial}{\partial g_3^1} + g_5^1\frac{\partial}{\partial g_5^1} + g_5^2\frac{\partial}{\partial g_5^2},\ \ Y_7 = -g_3^1\frac{\partial}{\partial g_3^1} + g_6^1\frac{\partial}{\partial g_6^1} + g_6^2\frac{\partial}{\partial g_6^2},\ Y_8 =\\
&= [1+(g_2^1)^2]\frac{\partial}{\partial g_2^1} + g_2^1 g_3^1\frac{\partial}{\partial g_3^1} - g_3^1 g_6^1\frac{\partial}{\partial g_5^1} + g_2^1 g_6^1\frac{\partial}{\partial g_6^1} - g_3^1 g_6^2\frac{\partial}{\partial g_5^2} + g_2^1 g_6^2\frac{\partial}{\partial g_6^2},
\end{aligned}
$$

and it acts transitively on the set of points $(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$.

From(3), applying (4), we obtain $g_6^1 - g_6^2 \neq 0$, and it is easy to verify that the infinitesimal operators Y_2, Y_3, Y_4, Y_5, Y_7, and Y_8 are arcwise unconnected, but $Y_6 = \lambda_3 Y_3 + \lambda_4 Y_4 + \lambda_5 Y_5$, where

$$
\lambda_3 = \frac{-g_5^1 g_6^2 + g_6^1 g_5^1}{g_6^1 - g_6^2},\ \lambda_4 = \frac{g_3^1(g_6^1 - g_6^2) + g_2^1(g_5^1 - g_5^2)}{g_6^1 - g_6^2},\ \lambda_5 = \frac{-g_5^1 + g_5^2}{g_6^1 - g_6^2}.
$$

Since $Y_3(\lambda_3) + Y_4(\lambda_4) + Y_5(\lambda_5) = 3 \neq 0$, we conclude that the following statement holds:

Theorem 2.1. *A set of pairs of parallel non-isotropic straight lines of the first type is not measurable with respect to the group G_8 and it has no measurable subsets.*

3. Measurability with respect to S_7. The associated group $\overline{S}_7$ of the group S_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$, and Y_7 from (6) and it acts transitively on the set of points $(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$. The integral invariant function $f = f(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$ satisfies the so-called system of R. Deltheil ([1, p.28]; [4, p.11]) $Y_1(f) = 0,\ Y_2(f) = 0,\ Y_3(f) = 0,\ Y_4(f) = 0,\ Y_5(f) = 0,\ Y_6(f) + 3f = 0,\ Y_7(f) + f = 0$, and has the form

$$
f = \frac{c}{(g_6^1 - g_6^2)[1 + (g_2^1)^2]^2},
$$

where $c = const$. Thus we establish the following

Theorem 3.1. *The set of pairs $(G_1, G_2)(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$ of parallel non-isotropic straight lines of the first type is measurable with respect to the group S_7 and has the density*

$$
(7)\qquad d(G_1, G_2) = \left|\frac{1}{(g_6^1 - g_6^2)[1 + (g_2^1)^2]^2}\right| dg_2^1 \wedge dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2 \wedge dg_6^2.
$$

Differentiating (4) and substituting into (7) we obtain another expression for the density:

Corollary 3.1. *The density (7) for the pairs (G_1, G_2) determined by the equations (2) can be written in the form*

$$
(8)\quad d(G_1, G_2) = \left|\frac{1}{[a(q_2 - q_1) - b(p_2 - p_1)](a^2 + b^2)^2}\right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.
$$

4. Measurability with respect to G_6. The associated group $\overline{G}_6$ of the group G_6 has the infinitesimal operators Y_1, Y_2, Y_3, Y_4, Y_7 from (6) and $Z = -3g_3^1\frac{\partial}{\partial g_3^1} - 2g_5^1\frac{\partial}{\partial g_5^1} +$

$g_6^1\dfrac{\partial}{\partial g_6^1} - 2g_5^2\dfrac{\partial}{\partial g_5^2} + g_6^2\dfrac{\partial}{\partial g_6^2}$. Since $\overline{G}_6$ acts intransitively on the set of points (g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2), the set of pairs (G_1, G_2) is not measurable with respect to G_6. The system $Y_1(f) = 0$, $Y_2(f) = 0$, $Y_3(f) = 0$, $Y_4(f) = 0$, $Y_7(f) = 0$, $Z(f) = 0$ has a solution $f = g_2^1$, and it is an absolute invariant of $\overline{G}_6$.

Consider the subset of pairs (G_1, G_2) satisfying the condition

$$g_2^1 = h, \tag{9}$$

where h =const. The group $\overline{G}_6$ induces the group $G_6^\star$ on the subset (9) with the infinitesimal operators Y_2, Y_3, Y_4, Y_7, Z and $U = g_3^1(\dfrac{\partial}{\partial g_5^1}+\dfrac{\partial}{\partial g_5^2})-h(\dfrac{\partial}{\partial g_6^1}+\dfrac{\partial}{\partial g_6^2})$, and it is transitive. The Deltheil system $Y_2(f) = 0, Y_3(f) = 0, Y_4(f) = 0, Y_7(f) + f = 0, Z(f) + 5f = 0, U(f) = 0$ has the solution $f = c(g_6^1 - g_6^2)^5$, where $c = const$.

From here it follows:

Theorem 4.1. *The set of pairs $(G_1, G_2)(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$ of parallel non-isotropic straight lines of the first type is not measurable with respect to the group G_6, but it has the measurable subset (9) with the density*

$$g(G_1, G_2) = |g_6^1 - g_6^2|^5 dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2 \wedge dg_6^2.$$

From Theorem 4.1. and (4) we have:

Corollary 4.1. *The set of pairs $(G_1, G_2)(a, b, p_1, q_1, p_2, q_2)$ of parallel non-isotropic straight lines of the first type is not measurable with respect to the group G_6, but it has the measurable subset $\dfrac{b}{a} = h, h = const$, with the density*

$$d(G_1, G_2) = \left|\frac{[q_2 - q_1 - h(p_2 - p_1)]^5}{a^4}\right| da \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.$$

5. Measurability with respect to B_7, W_7, G_7, V_7, B_6, B_5. By arguments similar to the ones used above we study the measurability of sets of pairs (G_1, G_2) with respect to all the remaining groups. We summarize the results in the following

Theorem 5.1. *The set of pairs $(G_1, G_2)(g_2^1, g_3^1, g_5^1, g_6^1, g_5^2, g_6^2)$ of parallel non-isotropic straight lines of first type:*

(i) is not measurable with respect to the groups B_7, G_7, B_6, and it has no measurable subsets;

(ii) is measurable with respect to the group W_7, and has the density

$$d(G_1, G_2) = \left|\frac{1}{(g_6^2 - g_6^2)^4\sqrt{1 + (g_2^1)^2}}\right| dg_2^1 \wedge dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2 \wedge dg_6^2;$$

(iii) is measurable with respect to the group V_7, and has the density

$$d(G_1, G_2) = \left|\frac{(g_6^1 - g_6^2)^5}{[1 + (g_2^1)^2]^5}\right| dg_2^1 \wedge dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2 \wedge dg_6^2;$$

(iv) is not measurable with respect to the group B_5, but it has the measurable subset $g_2^1 = h_1, g_6^1 - g_6^2 = h_2, h_1 = const, h_2 = const$, with the density

$$d(G_1, G_2) = dg_3^1 \wedge dg_5^1 \wedge dg_6^1 \wedge dg_5^2.$$

From Theorem 5.1. and (4) it follows

Corollary 5.1. *The set of pairs* $(G_1, G_2)(a, b, p_1, q_1, p_2, q_2)$ *of parallel non-isotropic straight lines of the first type:*

(i) is measurable with respect to the group W_7*, and has the density*

$$d(G_1, G_2) = \left| \frac{1}{[a(q_2 - q_1) - b(p_2 - p_1)]^4 \sqrt{a^2 + b^2}} \right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2;$$

(ii) is measurable with respect to the group V_7*, and has the density*

$$d(G_1, G_2) = \left| \frac{[a(q_2 - q_1) - b(p_2 - p_1)]^5}{(a^2 + b^2)^5} \right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2;$$

(iii) is not measurable with respect to the group B_5*, but it has the measurable subset* $\frac{b}{a} = h_1, a(q_2 - q_1) - b(p_2 - p_1) = h_2, h_1 = const, h_2 = const$*, with the density*

$$d(G_1, G_2) = \frac{1}{a^4}\, da \wedge dp_1 \wedge dq_1 \wedge dp_2.$$

6. Some Crofton type formulas with respect to S_7. The parallel straight lines into coordinate plane $\widetilde{G}_1 : bx - ay + aq_1 - bp_1 = 0, z = 0$, and $\widetilde{G}_2 : bx - ay + aq_2 - bp_2 = 0, z = 0$, are the orthogonal projections of the parallel straight lenes G_1 and G_2, respectively. Then the Euclidean distance δ between $\widetilde{G}_1$ and $\widetilde{G}_2$ is

$$\delta = \left| \frac{a(q_2 - q_1) - b(p_2 - p_1}{\sqrt{a^2 + b^2}} \right|. \tag{10}$$

Assume that the straight lines G_1 and G_2 make the angle θ with the horizontal plane Oxy. Then [3; p. 48]

$$\theta = \frac{1}{\sqrt{a^2 + b^2}} \tag{11}$$

and, replacing (10) and (11) into (8), we obtain

$$d(G_1, G_2) = \left| \frac{\theta^5}{\delta} \right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \tag{12}$$

If we denote $\overline{P}_1 = G_1 \cap Oxy$, $\overline{P}_2 = G_2 \cap Oxy$, then into the plane Oxy we have $d\overline{P}_1 = dp_1 \wedge dq_1$, $d\overline{P}_2 = dp_2 \wedge dq_2$. By differentiation of (10), (11), and by exterior multiplication of the forms of (12) we get

$$d\delta \wedge d\theta \wedge d\overline{P}_1 \wedge d\overline{P}_2 = -\theta^4 [a(q_2 - q_1) - b(p_2 - p_1]\, da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \tag{13}$$

In view of (13), the formula (12) becomes

$$d(G_1, G_2) = \left| \frac{\theta}{\delta[a(q_2 - q_1) - b(p_2 - p_1]} \right| d\delta \wedge d\theta \wedge d\overline{P}_1 \wedge d\overline{P}_2.$$

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ИЗМЕРИМОСТ НА МНОЖЕСТВА ОТ ДВОЙКИ УСПОРЕДНИ НЕИЗОТРОПНИ ПРАВИ ОТ ПЪРВИ ТИП В ПРОСТО ИЗОТРОПНО ПРОСРТАНСТВО

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Изследвана е измеримостта на множества от двойки успоредни неизотропни прави, лежащи в различни изотропни равнини и съответните инвариантни гъстоти относно групата на общите просто-изотропни подобности и някои нейни подгрупи.