МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2006
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2006
*Proceedings of the Thirty Fifth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 5–8, 2006*

ON THE EXPONENTS OF SOME 4×4 MATRICES*

Georgi K. Dimitrov, Ivaïlo M. Mladenov

Here we derive a formula for the exponents of any 4×4 matrix which belongs to one of the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(2,2)$, $\mathfrak{so}(3,1)$ and $\mathfrak{sp}(4,\mathbb{R})$. The approach which we follow is based on the Hamilton-Cayley theorem, namely, the important moment in all our considerations is the respective characteristic polynomial of the above matrices.

1. Introduction. This article concerns all 4×4 matrices with characteristic polynomial $P(z)$ of the form

$$P(z) = z^4 - bz^2 - a \tag{1}$$

which is shared by any element in the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$ and $\mathfrak{sp}(4,\mathbb{R})$. Both coefficients a and b in (1) are well defined functions of the respective matrix elements.

Let us denote with A an arbitrary matrix from the above classes. By the Hamilton-Cayley theorem it follows that A satisfies the identity

$$A^4 = a\mathrm{I}_4 + bA^2. \tag{2}$$

Direct consequence of the above equation and the very definition of the exponential map

$$\mathrm{Exp}(A) = \mathrm{I}_4 + \sum_{k=1}^{\infty} \frac{A^k}{k!} \tag{3}$$

is that

$$\mathrm{Exp}(A) = \mathrm{I}_4 + A + \frac{A^2}{2} + \frac{A^3}{6} \tag{4}$$

when $a = b = 0$.

2. Non-degenerated cases. From now on we exclude the degenerate case in which both coefficients are equal to zero. In order to derive the formula for the cases when either $a \neq 0$, or $b \neq 0$, we use (2) to get some quite useful relations about the even powers of A. Let us start by rewriting the equation (2) in the form

$$A^4 = uv\mathrm{I}_4 + (u - v)A^2, \tag{5}$$

*Partially supported by the Bulgarian National Science Foundation under a contract # B-1531/2005

where u and v are new parameters, which, obviously, have to satisfy the system

$$u - v = b, \qquad uv = a. \tag{6}$$

The solutions of this system are

$$u = \frac{1}{2}(b + \sqrt{b^2 + 4a}) \quad \text{or} \quad u = \frac{1}{2}(b - \sqrt{b^2 + 4a}) \quad \text{and} \quad v = \frac{a}{u}\cdot \tag{7}$$

From (6) it follows also that

$$(u + v)^2 = b^2 + 4a. \tag{8}$$

In fact, we derive two kind of formulas: one for the case when $b^2 + 4a \neq 0$, and another one for the case when $b^2 + 4a = 0$.

First case: $b^2 + 4a \neq 0$. In this case (8) ensure, that

$$u + v \neq 0. \tag{9}$$

Multiplying both sides of (5) with $u + v$, we get

$$(u + v)A^4 = (u + v)uv\mathrm{I}_4 + (u^2 - v^2)A^2. \tag{10}$$

Let us assume now, that for all $n \in \mathbb{N}$, $n > 2$ we have also

$$(u + v)A^{2n} = (u^{n-1} + (-1)^n v^{n-1})uv\mathrm{I}_4 + (u^n + (-1)^{n+1}v^n)A^2. \tag{11}$$

This equation, in conjuction with (9) and (10), gives

$$\begin{aligned}(u + v)A^{2(n+1)} =& (u^{n-1} + (-1)^n v^{n-1})uvA^2 + (u^n + (-1)^{n+1}v^n)A^4 \\ =& (u^{n-1} + (-1)^n v^{n-1})uvA^2 + (u^n + (-1)^{n+1}v^n)\big(uv\mathrm{I}_4 + (u - v)A^2\big) \\ =& (u^n + (-1)^{n+1}v^n)uv\mathrm{I}_4 + (u^{n+1} + (-1)^{n+2}v^{n+1})A^2.\end{aligned} \tag{12}$$

In this way by the full induction method we prove that (11) is really true for each $n \geq 0$. Actually, we proved this for $n \geq 2$, but one can verify it easily for $n = 0$ and $n = 1$ and, therefore, we have immediately

$$\begin{aligned}(u + v)\sum_{n=0}^{\infty}\frac{A^{2n}}{(2n)!} &= \left(v\sum_{n=0}^{\infty}\frac{u^n}{(2n)!} + u\sum_{n=0}^{\infty}\frac{(-1)^n v^n}{(2n)!}\right)\mathrm{I}_4 \\ &\quad + \left(\sum_{n=0}^{\infty}\frac{u^n}{(2n)!} - \sum_{n=0}^{\infty}\frac{(-1)^n v^n}{(2n)!}\right)A^2 \\ &= \big(v\cosh\sqrt{u} + u\cos\sqrt{v}\big)\,\mathrm{I}_4 + \big(\cosh\sqrt{u} - \cos\sqrt{v}\big)\,A^2.\end{aligned} \tag{13}$$

We can use again (11) to calculate the following sum

$$
\begin{aligned}
(u+v)\sum_{n=0}^{\infty}\frac{A^{2n+1}}{(2n+1)!} &= \left(v\sum_{n=0}^{\infty}\frac{u^n}{(2n+1)!}+u\sum_{n=0}^{\infty}\frac{(-1)^n v^n}{(2n+1)!}\right)A \\
&\quad+\left(\sum_{n=0}^{\infty}\frac{u^n}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{(-1)^n v^n}{(2n+1)!}\right)A^3 \\
&= \left(\frac{v}{\sqrt{u}}\sum_{n=0}^{\infty}\frac{\sqrt{u}^{2n+1}}{(2n+1)!}+\frac{u}{\sqrt{v}}\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{v}^{2n+1}}{(2n+1)!}\right)A \\
&\quad+\left(\frac{1}{\sqrt{u}}\sum_{n=0}^{\infty}\frac{\sqrt{u}^{2n+1}}{(2n+1)!}-\frac{1}{\sqrt{v}}\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{v}^{2n+1}}{(2n+1)!}\right)A^3 \\
&= \left(v\frac{\sinh\sqrt{u}}{\sqrt{u}}+u\frac{\sin\sqrt{v}}{\sqrt{v}}\right)A+\left(\frac{\sinh\sqrt{u}}{\sqrt{u}}-\frac{\sin\sqrt{v}}{\sqrt{v}}\right)A^3.
\end{aligned}
\tag{14}
$$

Let us remark also that the formula just derived is still valid even when u or v vanishes (in this case we take the limits of the functions $\frac{\sin(x)}{x}$ and $\frac{\sinh(x)}{x}$ as x tends to zero). Introducing

$$
\begin{aligned}
f_0(u,v) &= \frac{v\cosh\sqrt{u}+u\cos\sqrt{v}}{u+v}, \qquad f_1(u,v)=\frac{v\frac{\sinh\sqrt{u}}{\sqrt{u}}+u\frac{\sin\sqrt{v}}{\sqrt{v}}}{u+v}, \\
f_2(u,v) &= \frac{\cosh\sqrt{u}-\cos\sqrt{v}}{u+v}, \qquad f_3(u,v)=\frac{\frac{\sinh\sqrt{u}}{\sqrt{u}}-\frac{\sin\sqrt{v}}{\sqrt{v}}}{u+v},
\end{aligned}
\tag{15}
$$

we give up (13) and (14) the form

$$
\sum_{n=0}^{\infty}\frac{A^{2n}}{(2n)!}=f_0(u,v)\mathrm{I}_4+f_2(u,v)A^2, \qquad \sum_{n=0}^{\infty}\frac{A^{2n+1}}{(2n+1)!}=f_1(u,v)A+f_3(u,v)A^3,
\tag{16}
$$

which implies

$$
\mathrm{Exp}(A)=f_0(u,v)\mathrm{I}_4+f_1(u,v)A+f_2(u,v)A^2+f_3(u,v)A^3.
\tag{17}
$$

<u>Second case:</u> $b^2+4a=0$. In this case we can write

$$
A^4-bA^2-a\mathrm{I}_4=\left(A^2-\frac{b}{2}\mathrm{I}_4\right)^2=\left(A-\sqrt{\frac{b}{2}}\mathrm{I}_4\right)^2\left(A+\sqrt{\frac{b}{2}}\mathrm{I}_4\right)^2=0
\tag{18}
$$

and after introducing

$$
\sqrt{\frac{b}{2}}=\rho,
\tag{19}
$$

rewrite it as

$$(A-\rho\mathrm{I}_4)^2(A+\rho\mathrm{I}_4)^2=(A+\rho\mathrm{I}_4)^2(A-\rho\mathrm{I}_4)^2=0. \tag{20}$$

With this equation at hand we get immediately

$$\begin{aligned}\mathrm{Exp}\,(A-\rho\mathrm{I}_4)(A+\rho\mathrm{I}_4)^2&=[\mathrm{I}_4+(A-\rho\mathrm{I}_4)](A+\rho\mathrm{I}_4)^2\\&=[A+(1-\rho)\,\mathrm{I}_4](A+\rho\mathrm{I}_4)^2.\end{aligned} \tag{21}$$

By the properties of the exponential map, i.e.

$$\mathrm{Exp}\,(A-\rho\mathrm{I}_4)=\mathrm{Exp}(A)\mathrm{Exp}\,(-\rho\mathrm{I}_4)=\exp(-\rho)\,\mathrm{Exp}(A) \tag{22}$$

and (21), it follows that

$$\mathrm{Exp}(A)\,(A+\rho\mathrm{I}_4)^2=\exp(\rho)\,[A+(1-\rho)\,\mathrm{I}_4](A+\rho\mathrm{I}_4)^2. \tag{23}$$

Similar considerations gives us

$$\mathrm{Exp}(A)\,(A-\rho\mathrm{I}_4)^2=\exp(-\rho)\,[A+(1+\rho)\,\mathrm{I}_4](A-\rho\mathrm{I}_4)^2, \tag{24}$$

respectively.

Subtracting the left-hand side of (24) from that one of (23), we get

$$4\rho\mathrm{Exp}(A)A=2\sqrt{2b}\mathrm{Exp}(A)A. \tag{25}$$

Since $\det(A)=-a$, and $a\neq 0$, one can conclude that A is invertible. This is enough for us in order to write

$$\begin{aligned}\mathrm{Exp}(A)\quad=\quad&\frac{1}{4\rho}A^{-1}\Big\{\exp(\rho)\,[A+(1-\rho)\,\mathrm{I}_4](A+\rho\mathrm{I}_4)^2\\&-\exp(-\rho)\,[A+(1+\rho)\,\mathrm{I}_4](A-\rho\mathrm{I}_4)^2\Big\}\end{aligned} \tag{26}$$

which can be given in a more compact form as

$$\mathrm{Exp}(A)=g_0(\rho)A^{-1}+g_1(\rho)\mathrm{I}_4+g_2(\rho)A+g_3(\rho)A^2, \tag{27}$$

where

$$\begin{aligned}g_0(\rho)&=\frac{\rho\,\mathrm{sh}\,\rho-\rho^2\,\mathrm{ch}\,\rho}{2},\qquad g_1(\rho)=\frac{2\,\mathrm{ch}\,\rho-\rho\,\mathrm{sh}\,\rho}{2}\\g_2(\rho)&=\frac{\mathrm{sh}\,\rho+\rho\,\mathrm{ch}\,\rho}{2\rho},\qquad g_3(\rho)=\frac{\mathrm{sh}\,\rho}{2\rho}.\end{aligned} \tag{28}$$

Another useful way in which the above result can be presented is

$$\mathrm{Exp}(A)\quad=\quad\frac{\mathrm{ch}\,\rho}{2}(A+2\mathrm{I}_3-\rho^2A^{-1})+\frac{\mathrm{sh}\,\rho}{2\rho}(A^2+A-\rho^2\mathrm{I}_3+\rho^2A^{-1}). \tag{29}$$

3. Specializations for various Lie algebras. In this section we present the respective parameters a and b *via* explicit formulas for coordinates of Lie algebra elements from the selected list of Lie algebras given at the beginning of the paper.

3.1. The parameters a and b for the Lie algebra $\mathfrak{so}(4)$. The standard form of an arbitrary element $A \in \mathfrak{so}(4)$ is

$$A = \left\{ \begin{bmatrix} 0 & -x_1 & x_2 & -x_4 \\ x_1 & 0 & -x_3 & -x_5 \\ -x_2 & x_3 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{bmatrix} ; x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R} \right\} \tag{30}$$

and, in accordance with (2), the parameters a and b can be determined by evaluating its characteristic polynomial $P(A)$. For that purpose we used the elegant procedure described in [4]. The analytical and especially the computational details can be found in [1] along the *Mathematica*® program code furnishing this task. The results is

$$\begin{aligned} a &= -\left(x_1 x_6 + x_2 x_5 + x_3 x_4\right)^2 \\ b &= -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2. \end{aligned} \tag{31}$$

3.2. The parameters a and b for the Lie algebra $\mathfrak{so}(3,1)$. We fixed a basis in which the elements of $\mathfrak{so}(3,1)$ are of the form

$$\mathfrak{so}(3,1) = \left\{ \begin{bmatrix} 0 & -x_1 & x_2 & x_4 \\ x_1 & 0 & -x_3 & x_5 \\ -x_2 & x_3 & 0 & x_6 \\ x_4 & x_5 & x_6 & 0 \end{bmatrix} ; x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R} \right\}, \tag{32}$$

while the parameters a and b in the above coordinates turns out to be presented by the following expressions

$$\begin{aligned} a &= \left(x_1 x_6 + x_2 x_5 + x_3 x_4\right)^2 \\ b &= -x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2. \end{aligned} \tag{33}$$

We take the opportunity to mention that any constant electromagnetic field can be described by a second order tensor of type (32) and that the trajectories of charged particles in such fields can be obtained by making use of its exponent [5]. We refer to [2] for details and graphics illustrating various physical situations.

3.3. The parameters a and b for the Lie algebra $\mathfrak{so}(2,2)$. Now we have

$$\mathfrak{so}(2,2) = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & x_4 \\ -x_1 & 0 & x_3 & x_5 \\ x_2 & x_3 & 0 & x_6 \\ x_4 & x_5 & -x_6 & 0 \end{bmatrix} ; x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R} \right\}. \tag{34}$$

and,

$$\begin{aligned} a &= -\left(x_1 x_6 + x_2 x_5 - x_3 x_4\right)^2 \\ b &= -x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2, \end{aligned} \tag{35}$$

respectively.

3.4. The parameters a and b for the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. Any element of this Lie algebra can be specified by means of the coordinates $x_i,\ i=1,\ldots,10$ and has the form

$$A=\left\{\begin{bmatrix} x_1 & x_2 & x_5 & x_6\\ x_3 & x_4 & x_6 & x_7\\ x_8 & x_9 & -x_1 & -x_3\\ x_9 & x_{10} & -x_2 & -x_4\end{bmatrix};\ x_1,\ldots,x_{10}\in\mathbb{R}\right\}. \tag{36}$$

Unfortuately, this time the parameters

$$\begin{aligned} a &= -x_1^2x_4^2 - x_2^2x_3^2 - x_6^2x_9^2 - x_1^2x_7x_{10} - x_3^2x_5x_{10} - x_2^2x_7x_8 - x_4^2x_5x_8\\ &\quad +x_6^2x_8x_{10} + x_9^2x_5x_7 + 2x_1x_2x_3x_4 - 2x_1x_4x_6x_9 + 2x_1x_2x_7x_9\\ &\quad +2x_1x_3x_6x_{10} + 2x_2x_4x_6x_8 - 2x_2x_3x_6x_9 + 2x_3x_4x_5x_9 - x_5x_7x_8x_{10}\\ b &= x_1^2 + x_4^2 + 2x_2x_3 + 2x_6x_9 + x_5x_8 + x_7x_{10}. \end{aligned} \tag{37}$$

are not so symmetrical as in preceding cases but are still manageable, especially for the purposes of the direct numerical implementations.

A *Mathematica*® program module which returns automatically the relevant matrix exponent by appropriate input matrices belonging to the classes discussed so far is available for testing and free use [3].

Acknowledgements. The authors are grateful to Dr. Clementina D. Mladenova from the Institute of Mechanics – Bulgarian Academy of Sciences for providing an access to her PC and computer algebra system *Mathematica*®. Most of the calculations presented here were done *via* above mentioned system.

REFERENCES

[1] G. Dimitrov G., I. Mladenov. `http://obzor.bio21.bas.bg/dpb_files/mfiles/Lev_Fad_Alg.nb`.

[2] G. Dimitrov G., I. Mladenov. A New Formula for the Exponents of the Generators of the Lorentz Group. In: Proceedings of the Seventh International Conference on Geometry, Integrability and Quantization, (Eds I. Mladenov, M. de León), SOFTEX, Sofia 2005.

[3] G. Dimitrov G., I. Mladenov. `http://obzor.bio21.bas.bg/dpb_files/mfiles/4x4Exp_Map.nb`.

[4] S.-H. Hou. A Simple Proof of the Leverrier-Faddeev Characteristic Polynomial Algorithm. *SIAM Rev.*, **40** (1998), 706–709.

[5] R. Zeni, W. Rodrigues. The Exponential of the Generators of the Lorentz Group and the Solution of the Lorentz Force. *Hadronic Journal*, **3** (1990), 317–327.

Georgi K. Dimitrov
Institute of Biophysics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 21
1113 Sofia, Bulgaria
e-mail: gkid@abv.bg

Ivaïlo M. Mladenov
Institute of Biophysics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 21
1113 Sofia, Bulgaria
e-mail: mladenov@obzor.bio21.bas.bg

ВЪРХУ ЕКСПОНЕНТИТЕ НА НЯКОИ 4×4 МАТРИЦИ

Георги К. Димитров, Ивайло М. Младенов

В настоящата работа е изведена обща формула за експонентите на всички 4×4 матрици принадлежащи на някоя от следните алгебри на Ли: $\mathfrak{so}(4)$, $\mathfrak{so}(2,2)$, $\mathfrak{so}(3,1)$ и $\mathfrak{sp}(4,\mathbb{R})$. Подходът за решаване на задачата се основава на теоремата на Хамилтон-Кейли и, по-точно, от съществено значение е видът на характеристичният полином на матриците на изброените по-горе алгебри на Ли.