МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2006
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2006
Proceedings of the Thirty Fifth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 5–8, 2006

SOME SELF-SIMILAR SETS DEFINED BY A PAIR OF PLANE CONTRACTING SIMILARITIES*

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We consider special pairs of plane contracting similarities with the property: each similarity maps the fixed point of the other similarity to one and the same point. We investigate the invariant sets of all such pair of similarities.

1. Preliminaries. A transformation $S:\mathbb{R}^2\to\mathbb{R}^2$ is called a contracting similarity with ratio $c\in(0,1)$ when $|S(z_1)-S(z_2)=c|z_1-z_2|$ for any $z_1,z_2\in\mathbb{R}^2$. A finite family of contracting similarities

$$S_1,\, S_2,\, \ldots,\, S_m \tag{1}$$

is called an iteration function system. We recall two basic facts for iteration function systems. The proofs are known from [4]. There is a unique non-empty compact set $F\subset\mathbb{R}^2$, called an attractor or an invariant set, such that $F=\bigcup_{i=1}^{m}S_i(F)$. The set F is, in general, a fractal. The standard procedure for obtaining of F is the following. Let $E\subset\mathbb{R}^2$ be an arbitrary non-empty compact set and $T(E)=\bigcup_{i=1}^{m}S_i(E)$. Consider the sequence $T^0(E)=E$, $T^1(E)=T(E)$, …, $T^{(k)}(E)=T(T^{(k-1)}(E))$, then $F=\bigcap_{k=0}^{\infty}T^{(k)}(E)$. The set F is often called self-similar. A systematic description of self-similar sets is given in [3] and [4].

If there are m+1 different points in $\mathbb{R}^2$ $p_0, p_1, \ldots, p_m$ such that $S_i(p_0)=p_{i-1}$, $S_i(p_m)=p_i$, for $i=1,\ldots,m$, the iterated function system is called a zipper with signature $(\sigma_1=0,\ldots,\sigma_m=0)$. Many properties of zippers are proved in [1] and [2].

In this paper we study zippers with the base points p_0, p_1 and p_2. This means that either $p_0p_1p_2$ is a triangle with acute angles $\sphericalangle p_1p_0p_2$ and $\sphericalangle p_1p_2p_0$, or p_1 is an interior point of the line segment $[p_0p_2]$.

2. A zipper defined by two direct similarities. Any orientation-preserving similarity, called also a direct similarity, $S:\mathbb{R}^2\to\mathbb{R}^2$ with fixed point (x_0,y_0) can be

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*Research partially supported by Shumen University under grant 24200305.
Key words: contracting similarities, iterated function system
2000 Mathematics Subject Classification: 51M15, 28A80.

represented by the matrix equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = c \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where $c \in \mathbb{R} \setminus \{0\}$ is the ratio of similarity and $\varphi \in [-\pi, \pi]$ is the angle of the rotation. This similarity is a contraction whenever $c \in (-1, 0) \cup (0, 1)$. We consider an iterated function system formed by two contracting similarities S_1 and S_2 which have different fixed points $(0, 0)$ and $(1, 0)$ respectively. Then the corresponding matrix equation of S_1 and S_2 are

$$S_1 : \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = c_1 \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

and

$$S_2 : \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = c_2 \begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix} \begin{pmatrix} x - 1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

where $c_i \in (0, 1)$ for $i = 1, 2$.

Proposition 1. *Let a and b be positive real numbers such that $a^2 + b^2 < 1$ and $(a-1)^2 + b^2 < 1$. Then the iterated function system $\{S_1, S_2\}$ is a zipper with vertices $p_0 = (0, 0)$, $p_1 = (a, b)$ and $p_2 = (1, 0)$ if and only if $c_1 = \sqrt{a^2 + b^2}$, $c_2 = \sqrt{(1-a)^2 + b^2}$, $\tan\varphi_1 = \dfrac{b}{a}$ and $\tan\varphi_2 = \dfrac{b}{a-1}$.*

Proof. Since the function tan is a bijection of the open interval $(-\pi/2, \pi/2)$, the conditions $S_1(p_2) = p_1$ and $S_2(p_0) = p_1$ are equivalent to the conditions $c_1 = \sqrt{a^2 + b^2}$, $c_2 = \sqrt{(1-a)^2 + b^2}$, $\varphi_1 = \arctan \dfrac{b}{a}$ and $\varphi_2 = \arctan \dfrac{b}{a-1}$. $\square$

Let F be the invariant set of the zipper $\{S_1, S_2\}$. According to Lemma 1.1 from [2] there is a structural parametrization of F. This means that there exists a continuous mapping $\gamma : [0, 1] \to F$ with the property: if $\bar{t} \in (0, 1)$, $\gamma(0, \bar{t}, 1) = (p_0, p_1, p_2)$, $\tau_1(t) = \bar{t}.t$ and $\tau_2(t) = \bar{t}.(1-t) + t$ for all $t \in [0, 1]$, then $S_i \circ \gamma = \gamma \circ \tau_i$ for $i = 1, 2$.
Without loss of generality we may assume that $\bar{t} : (1 - \bar{t}) = c_1 : c_2$.

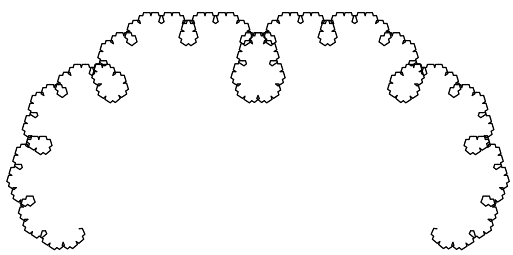

Fig. 1. $p_1 = (1/2, \sqrt{5}/6)$

Corollary 1. *The invariant set F is not a Jordan arc.*

Proof. It is sufficient to show that γ is not a homeomorphism. From (2) and (3) it follows that the sequence of points $S_1 \circ S_2^k(p_1) \in F$, $k = 1, 2, \ldots$, tends to $p_1 \in F$. The second sequence $S_2 \circ S_1^k(p_1)$, $k = 1, 2, \ldots$, also tends to p_1. Hence, p_1 is a self-intersecting point of F, i.e γ is not a homeomorphism (see Figure 1).

3. A zipper defined by a pair of direct and orientation-reversing similarities. A contracting orientation-reversing similarity $\widetilde{S}_2 : \mathbb{R}^2 \to \mathbb{R}^2$ with fixed point $(1, 0)$ can be represented in the form

$$\widetilde{S}_2 : \quad \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \widetilde{c}_2 \begin{pmatrix} \cos\varphi_2 & \sin\varphi_2 \\ \sin\varphi_2 & -\cos\varphi_2 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4}$$

where $\widetilde{c}_2 \in (0, 1)$ and $\varphi_2 \in [-\pi, \pi]$. As in the previous section we obtain:

Proposition 2. *Let $a > 0$, $b > 0$, $a^2 + b^2 < 1$ and $(1-a)^2 + b^2 < 1$. Suppose that $S_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by (2) and $\widetilde{S}_2$ is given by (4). Then, the iterated function system $\{S_1, \widetilde{S}_2\}$ is a zipper with vertices $p_0 = (0, 0)$, $p_1 = (a, b)$ and $p_2 = (1, 0)$ if and only if $c_1 = \sqrt{a^2 + b^2}$, $c_2 = \sqrt{(1-a)^2 + b^2}$, $\tan\varphi_1 = \dfrac{b}{a}$ and $\tan\varphi_2 = \dfrac{b}{1-a}$.*

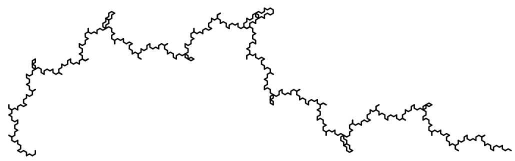

Fig. 2. $p_1 = (1/2, \sqrt{3}/6)$

Corollary *The invariant set F of $\{S_1, \widetilde{S}_2\}$ is not a Jordan arc.*

Proof. Let E be the unit segment with endpoints $p_0 = (0, 0)$ and $p_2 = (1, 0)$. For $k \geq \dfrac{\pi - \varphi_1 - \varphi_2}{\varphi_1}$ we have that $Card(S_1(T^{k-1}(E)) \cap S_2(T^{k-1}(E))) > 1$. Since $T^k(E) \to F$ as $k \to \infty$, where F (see Figure 2) is the attractor of $\{S_1, \widetilde{S}_2\}$, $Card(S_1(F) \cap S_2(F)) > 1$. Now, let γ be the structural parametrization of F. Assume that F is a Jordan arc. This means that γ is a homeomorphism. From $Card(S_1(F) \cap S_2(F)) > 1$ it follows that there exist $F \ni x \neq p_1 = (a, b)$ and $x \in S_1(F) \cap S_2(F)$. Hence, $\gamma^{-1}(x) \in \tau_1([0, 1]) = [0, \bar{t}]$ and $\gamma^{-1}(x) \in \tau_2([0, 1]) = [\bar{t}, 1]$, i. e. $\gamma^{-1}(x) = \bar{t}$. Therefore, $x = p_1$ which is a contradiction. $\square$

4. Zippers defined by two orientation-reversing similarities. Using complex numbers for representation of contracting similarities of $\mathbb{R}^2 \cong \mathbb{C}$ we can obtain general conditions for zipper with signature (0,0) and vertices $p_0 = 0 \in \mathbb{C}$, $\quad p_1 = a + bi = \alpha \in \mathbb{C}$ and $p_2 = 1 \in \mathbb{C}$. It is well known that any direct similarity of $\mathbb{R}^2 \cong \mathbb{C}$ is given by the equation $S(z) = qz + r$; $q \in \mathbb{C} \setminus \{0\}$, $r, z \in \mathbb{C}$ and any orientation-reversing similarity of $\mathbb{R}^2 \cong \mathbb{C}$ is given by $S(z) = q\overline{z} + r$; $q \in \mathbb{C} \setminus \{0\}$, $r, z \in \mathbb{C}$.

Theorem 1. *Let $\alpha \in \mathbb{C}$, $\quad |\alpha| < 1$ and $|1 - \alpha| < 1$. Let $\mathbf{S} = \{S_1, S_2\}$ be a pair of contracting similarities of $\mathbb{R}^2 \cong \mathbb{C}$, i.e. for $i \in \{1, 2\}$ either $S_i(z) = q_i z + r_i$, $|q_i| < 1$, or $S_i(z) = q_i\overline{z} + r_i$, $|q_i| < 1$. Then, the iterated function system $\mathbf{S} = \{S_1, S_2\}$ is a zipper with signature (0,0) and vertices at the points $\{0, \alpha, 1\}$ if and only if $r_1 = 0$, $\ r_2 = \alpha$, $\ q_1 = \alpha$ and $q_2 = 1 - \alpha$.*

Proof. From the definition of a zipper (see [1]) the iterated function system $\mathbf{S} = \{S_1, S_2\}$ is a zipper with signature (0,0) whenever

$$S_1(0) = 0,\ S_2(0) = \alpha \ \text{ and} \tag{5}$$

(6) $$S_1(1) = \alpha,\ S_2(1) = 1.$$

Applying (5) we get $r_1 = 0,\ r_2 = \alpha$. Then using (6) we obtain $q_1 = \alpha$ and $q_2 = 1-\alpha$. $\square$

Obviously, if $\alpha \in \mathbb{R}$, or more precisely $\alpha \in (0,1)$, then the the attractor F of $\mathbf{S}$ is the unit segment. So that in the remaining part of the paper we can assume that $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

Now we shall investigate the zippers $\mathbf{S} = \{S_1,\ S_2\}$ of two orientation-reversing contracting similarities of the Euclidean plane. From **Theorem 1** we have that $S_1(z) = \alpha\overline{z}$, $S_2(z) = (1-\alpha)\overline{z} + \alpha$, where $|\alpha| < 1$, $|1-\alpha| < 1$, $\alpha,\ z \in \mathbb{C}$ and $\mathbf{S}$ has vertices at the points $\{0,\ \alpha,\ 1\}$.

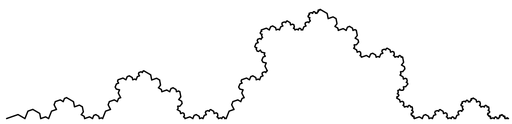

Fig. 3. $\alpha = 5/8 + i\sqrt{3}/8$

Corollary 3. *The invariant set F of of the zipper $\mathbf{S}$ defined by two orientation-reversing similarities is a Jordan arc if and only if $|\alpha - 1/2| < 1/2$.*

Proof. F is a Jordan arc with endpoints 0 and 1 if $Card(S_1(F) \cap S_2(F)) = 1$ (see [2], Theorem 1.2). If E is the unit segment with endpoints 0 and 1, then $T^1(E)$ is union of two segments with endpoints 0, 1 and a common endpoint α. At the next iteration we obtain the points $S_1(\alpha)$ and $S_2(\alpha)$ which belong to the segment E since S_i changes the orientation. The distance between the points $S_1(\alpha)$ and $S_2(\alpha)$ is $1 - |\alpha|^2 - |1-\alpha|^2$. Continuing in this way we get 2^k new points at the k-th iteration and the distances between the points obtained by the same new point at $(k-1)$-th iteration are $|\alpha|^{k-2-s}|1-\alpha|^s(1 - |\alpha|^2 - |1-\alpha|^2)$, $s = 0, 1, \ldots, k-2$, $k \geq 2$. Hence, $Card(S_1(F) \cap S_2(F)) = 1$ if and only if $1 - |\alpha|^2 - |1-\alpha|^2 > 0$. The last inequality is equivalent to $|\alpha - 1/2| < 1/2$ and this completes the proof see (Figure 3).

The sets $T^k(E)$ are called pre-fractals of F.

Proposition 3. *Let the Jordan arc F be the attractor of $\mathbf{S}$. Then the pre-fractal $T^k(E)$ has a length $(|\alpha| + |1-\alpha|)^k$.*

Proof. We use the principle of complete induction. It is clear that the statement is true for $k = 1$. Moreover, $|\alpha| + |1-\alpha| > 1$. We suppose that the length of the pre-fractal $T^{k-1}(E)$ is $(|\alpha| + |1-\alpha|)^{k-1}$. Since $Card(S_1(T^{k-1}(E)) \cap S_2(T^{k-1}(E))) = 1$ the length of $T^k(E)$ is $|\alpha|(|\alpha| + |1-\alpha|)^{k-1} + |1-\alpha|(|\alpha| + |1-\alpha|)^{k-1} = (|\alpha| + |1-\alpha|)^k$. $\square$

If $c_i \in (0,\, 1)$, $i = 1, \ldots, m$ are the ratios of the similarities S_i of the system (1), then the unique solution s of the equation $\sum_{i=1}^{m} c_i^s = 1$ is called a similarity dimension. In Euclidean space the similarity dimension and the Hausdorff dimension $\dim_{\mathcal{H}} F$ of F coincides if the "pieces" $S_i(F)$ are pairwise disjoint. This result remains true if the pieces have only "small overlap", so called "just touching" (see [5]). Consequently, if the attractor F of the zipper $\mathbf{S} = \{S_1,\ S_2\}$ is a Jordan arc, then $\dim_{\mathcal{H}} F = s$, where s the solution of the equation $|\alpha|^s + |1-\alpha|^s = 1$.

Summarizing the results of this and previous sections we get the main theorem in this paper.

Theorem 2. *Let S_1 and S_2 be two contracting similarities of $\mathbb{R}^2 \cong \mathbb{C}$ and let $\mathbf{S} = \{S_1, S_2\}$ be a zipper with signature $(0, 0)$ and vertices $\{0, \alpha, 1\}$. Then the attractor F of $\mathbf{S}$ is a Jordan arc if and only if S_1 and S_2 are orientation-reversing similarities and $\alpha \in \mathbb{C} \setminus \mathbb{R}$, $|\alpha - 1/2| < 1/2$.*

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НЯКОИ САМОПОДОБНИ МНОЖЕСТВА, ОПРЕДЕЛЕНИ ЧРЕЗ ДВОЙКА РАВНИННИ СВИВАЩИ ПОДОБНОСТИ

Радостина П. Енчева, Георги Хр. Георгиев

Разглеждат се специални двойки от равнинни свиващи подобности със свойството: всяка подобност изпраща неподвижната точка на другата в една и съща фиксирана точка. Изучават се инвариантните множества на всички такива двойки подобности.