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# EXISTENCE OF SOLUTION OF THE CAUCHY PROBLEM FOR SEMILINEAR HEAT EQUATIONS

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In this paper we study the Cauchy problem of semilinear parabolic equation $\partial_t u - \Delta u + v(x)u = F(t,x,u)$, where $v(x) \neq 0$. An estimate on lifespan for nonlinearities of the type $|F(t,x,\lambda)| \leq \text{const.}\lambda^{1+\alpha}$, $\alpha > 0$ is found. In cases when the solution blows up are found sufficiently general conditions on the impulsive sources which lead to global existence of solution of the impulsive parabolic Cauchy problem.

**1. Introduction.** In this paper we study the existence of solution of the Cauchy problem

$$\mathcal{L}_v(u) = F(t,x,u) \quad \text{in} \quad S_T = (0,T) \times \mathbb{R}^n \tag{1}$$

$$u(0,x) = f(x) \quad \text{on} \quad \mathbb{R}^n \tag{2}$$

where $\mathcal{L}_v(u) = \partial_t u - \Delta u + v(x)u$, $\partial_t = \dfrac{\partial}{\partial t}$, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$, $0 < T \leq \infty$ and $F$, $f$ and the potential $v$ are given functions.

There are many papers which treat the case $v(x) = 0$ and $F(t,x,\lambda) = \lambda^{1+\alpha}$. The results of Fujita [1] and Weissler [5] show that the solution exists if $\alpha > \frac{2}{n}$. If $\alpha \leq \frac{2}{n}$. then the solution blows up for a finite time (see also Samarski et al. [6]).

More general nonlinearities $F$, including $\nabla u$, are considered by Klainerman [2], Ponce [3]. In M. L. Marinov, V. S. Georgiev [7] is treated the case $v(x) \neq 0$. In the case $v(x) \geq 0$ it is proved global existence of solution of the Cauchy problem with small initial condition, when $|F(t,x,\lambda)| \leq C|\lambda|^{1+\alpha}$, $\alpha > \frac{2}{n}$.

The study of impulsive partial differential equations started recently by a paper of Erbe, Freedman, Liu an Wu [8]. Bainov, Kolev and Nakagawa [9] considered initial boundary value problem for semilinear parabolic equations with impulsive effect. In the case when the initial data are not too small they investigate how to control the impulsive source to delay the blowing-up time $T^*$.

In this paper we obtain:

1) The estimate on time of local existence of solution of the Cauchy problem.

2) Sufficiently general condition on the impulsive source which leads to global solution of the impulsive Cauchy problem.

3) The estimate on the lifespan of solution for arbitrary potentials $v(x)$.

**2. Notations and main assumptions.** We denote by $X$ the Banach space $L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$, with norm $|\cdot|_X = \|\cdot\|_{L^1(\mathbb{R}^n)} + \|\cdot\|_{L^\infty(\mathbb{R}^n)}$. Here $C_0(\mathbb{R}^n)$ is the closure of functions

$f \in C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|f\|_\infty = \sup\limits_{x\in\mathbb{R}^n} |f(x)|$. The space $C_0(\mathbb{R}^n)$ consists of all continuous functions which tend to zero at infinity.

For any $t > 0$ we define $\mathcal{U}_0(t): \ X \to X$ with

$$\mathcal{U}_0(f)(t,x) = \int\limits_{\mathbb{R}^n} \Gamma(t, x-y) f(y) dy$$

where $\Gamma(t,x) = (4\pi t)^{-\frac{n}{2}} \exp\left\{\dfrac{-|x|^2}{4t}\right\}$.

In addition we introduce the following notations: $\mathbb{R}_+ = (0,\infty)$; $Q(T) = (0,T)\times\mathbb{R}^n$, $0 < T \le \infty$; $C(0,T;X)$ is the Banach space of all continuous functions $u: \ [0,T] \to X$ with norm $|u|_C = \sup\limits_{[0,T]} |u(t)|_X$.

Let $0 = T_0 < T_1 < T_1 < \cdots T_k < T_{k+1} < \cdots$, and $\lim\limits_{k\to\infty} T_k = \infty$. We define the following sets: $P_k = \{(T_k, x) \colon x \in \mathbb{R}^n\}$; $P = \bigcup\limits_{k=1}^{\infty} P_k$; $\mathring{\mathbb{R}} = \mathbb{R}_+ \setminus \{T_k\}$ and $C(\mathring{\mathbb{R}}; X)$ is the set of all functions $u$: $\bar{R}_+ \to X$ which satisfy the conditions

(i) $u(t,\cdot) \in C(T_{k-1}, T_k; X)$, $k \in N = \{1, 2, \dots\}$;

(ii) $\lim\limits_{t \to T_k-} u(t,x) = u(T_k-, x) \in X$, $k \in N$.

Now we formulate the main assumptions.

$(H1)$ The potential $v(x)$ is bounded, Hölder continuous function.

$(H2)$ $\left\{\begin{array}{l}\text{There exist suitable positive constants } a,\ \alpha \text{ and } C \text{ such that } F(t,x,\lambda) \in \\ C^1(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R});\ |F(t,x,\lambda)| \le C|\lambda|^{1+\alpha} \text{ and } |F'_\lambda(t,x,\lambda)| \le C,\ \forall\ (t,x,\lambda) \in \\ [0,\infty)\times\mathbb{R}^n\times[-a,a].\end{array}\right.$

**3. Main result.**

**Theorem 1.** *Suppose that the assumptions $(H1)$ and $(H2)$ are fulfilled. Then for any $f \in X$ with $|f|_X < \dfrac{a}{2}$ the solution of the Cauchy problem*

$$(3) \qquad \left\{\begin{array}{lrl} \text{(a)} & \partial_t u - \Delta u + v(x)u & = F(t,x,u) \quad in \quad Q(T) \\ \text{(b)} & u(0,x) & = f(x) \end{array}\right.$$

*exists and $u \in C(0,\tau;X)$ for*

$$\tau = \min\left\{\frac{1}{2}[\|v\|_\infty + M(a)a^\alpha]^{-1}, \frac{1}{2}[\|v\|_\infty + M(a)]^{-1}\right\},$$

*where*

$$M(s) = \sup_{0<|\lambda|\le s}\left\{|\lambda|^{-1-\alpha}\sup_{(t,x)}\{|F(t,x,\lambda)|\}\right\} + \sup_{0<|\lambda|\le s}\left\{\sup_{(t,x)}\{|F'_\lambda(t,x,\lambda)|\}\right\}$$

*If $|f|_X < \eta$, $0 < \eta \le \dfrac{1}{2}a$, then $u(t,x)$ satisfies the following estimates*

$$(4) \qquad \left\{\begin{array}{l} \text{(a)} \ \sup\limits_{[0,\tau]}\{|u(t,\cdot) - \mathcal{U}_0(f)|_X\} \le \eta \\ \\ \text{(b)} \ |u(t,\cdot)|_X \le 2\eta, \quad \forall\ t \in [0,\tau]. \end{array}\right.$$

**Proof.** The function $\mathcal{U}_0(f) \in C(0,\infty;X)$ and is the classical solution of the problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } Q(\infty) \\ u(0,x) = f(x) & \text{on } \mathbb{R}^n \end{cases}$$

(see [4]). Using the fact that $e^{t\Delta} = \mathcal{U}^0$ is a contraction semigroup in the class $C_0$, and also using the Duhamel's principal (see [10]), we verify, that the problem

$$\begin{cases} \partial_t u - \Delta u = F_0(t,x) & \text{in } Q(\infty) \\ u(0,x) = f(x) & \text{on } \mathbb{R}^n, \end{cases}$$

for $f(x) \in X$ and $F_0 \in C(0,\infty;X)$ has the unique solution

$$u(t,x) = \mathcal{U}_0(f)(t,x) + \int_0^t \mathcal{U}_0(F_0(s,\cdot))(t-s,x)ds,$$

which belongs to $C(0,\infty;X)$. Besides that, the solution $u(t,x)$ satisfies the estimate

$$|u(t,\cdot) - \mathcal{U}_0(f)(t,\cdot)|_X \leq t \sup_{0\leq s\leq t} \{|F_0(s,\cdot)|_X\}. \tag{5}$$

This allows us to solve the problem (3) by using the contraction principle. We study the Cauchy problem via the corresponding integral equation $u = J(u)$, where

$$J(u) = \mathcal{U}_0(f)(t,x) + \int_0^t \mathcal{U}_0(F_v(s,\cdot,u(s,\cdot))(t-s,x)ds \tag{6}$$

and

$$F_v(s,y,u(s,y)) = -v(y)u(s,y) + F(s,y,u(s,y)).$$

For arbitrary $\eta > 0$ we define the set

$$B(\eta) = \{u \in C(0,\tau;X) : \sup_{0\leq t\leq \tau} \{|u(t,\cdot) - \mathcal{U}_0(f)(t,\cdot)|_X \leq \eta\}.$$

We point out that $B(\eta)$ is convex and closed subset of the Banach space $C(0,\tau;X)$.

We will prove that if $0 < \eta \leq \dfrac{a}{2}$, we have

$$J(B(\eta)) \subset B(\eta).$$

Let $u \in B(\eta)$. Then using again that $e^{t\Delta}$ is a contraction semigroup we prove:

$$|u(t,\cdot)|_X \leq |u(t,\cdot) - \mathcal{U}_0(f)(t,\cdot)|_X + |\mathcal{U}_0(f)(t,\cdot)|_X \leq \eta + |f|_X$$

Hence, if $|f|_X < \eta$, then

$$|u(t,\cdot)|_X < 2\eta \leq a, \quad \forall\ t \in (0,\tau]. \tag{7}$$

Thus, $\|u(t,\cdot)\|_{L^\infty_x} \leq |u(t,\cdot)|_X < a$ and it follows from the condition $(H2)$ that

$$|F_v(s,\cdot,u(s,\cdot))|_X \leq (\|v\|_\infty + M(a)a^\alpha)|u(s,\cdot)|_X,\ \forall\ s \in [0,\tau]. \tag{8}$$

From the inequalities (5), (8) и (7) we get that $\forall\ t \in [0,\tau]$

$$|J(u)(t,\cdot) - \mathcal{U}_0(f)(t,\cdot)|_X \leq \tau(\|v\|_\infty + M(a)a^\alpha)2\eta \leq \eta$$

and therefore $J(u)(t,x) \in B(\eta)$.

To apply the contraction principle we must estimate the norm of

$$J(u_1) - J(u_2) = \int_0^t \mathcal{U}_0(\Phi(u_1,u_2))(t-s,x)ds,$$

where

$$\Phi(u_1,u_2)(s,y)=v(y)[u_2(s,y)-u_1(s,y)]+F(s,y,u_1(s,y))-F(s,y,u_2(s,y)).$$

Once again, using that the semigroup $e^{t\Delta}$ is contraction and similarly to the inequality (5), we get

$$\begin{aligned}|J(u_1)(t,\cdot)-J(u_2)(t,\cdot)|_X &\le t\sup_{0\le s\le t}|\Phi(u_1,u_2)(s,\cdot)|_X\le\\ &\le \tau\sup_{0\le s\le \tau}\{(\|v\|_\infty+M(a))|u_1(s,\cdot)-u_2(s,\cdot)|_X\}\le\\ &\le \tau(\|v\|_\infty+M(a))|u_1-u_2|_C\le\frac{1}{2}|u_1-u_2|_C.\end{aligned}$$

Then

$$|J(u_1)-J(u_2)|_C\le\frac{1}{2}|u_1-u_2|_C$$

and the contraction principle is applicable.

Hence, in $B(\eta)$ there exists a unique solution on the equation $u=J(u)$, which apparently satisfies (4a) and because of the inequality (7) the inequality (4b) is satisfied. This completes the proof of Theorem 1. □

**Theorem 2.** *Assume that conditions* $(H1)$ *and* $(H2)$ *hold and the positive constant* $\tau$ *is defined in Theorem 1. Then for any* $f\in X$ *with* $|f|_X\le 2^{-k}a$, $k\in N$, *the Cauchy problem* (3) *has a unique solution* $u\in C(0,k\tau;X)$. *The solution* $u(t,x)$ *satisfies the estimate*

$$|u(t,\cdot)|_X\le 2^{-k+i}a,\qquad \forall\ t\in((i-1)\tau,i\tau],\ \forall\ i\in\{1,2,\dots,k\}.$$

**Proof.** Let $f\in X$ and $|f|_X\le 2^{-k}a$ for some $k\in\mathbb{N}$. By Theorem 1 we have unique solution $u_0\in C(0,\tau;X)$, to the Cauchy problem (3) with initial data $u_0(0,x)=f(x)$. The solution $u_0(t,x)$ satisfies the estimate

$$|u_0(t,.)|_X\le 2^{-k+1}a,\quad t\in[0,\tau].$$

Now, let $u_i\in C(0,\tau;X)$ be the solution of the Cauchy problem (3) with initial data $u_i(0,x)=f_i(x)$, where $f_i(x)=u_{i-1}(\tau,x)$, $i\in\{1,\dots,k-1\}$.

From Theorem 1 we get

$$\begin{array}{ll}|u_{i-1}(t,.)|_X\le 2^{-k+i}a, & t\in(0,\tau]\ \text{and}\\ |f_i|_X\le 2^{-k+i}a, & i\in\{1,2,\dots,k\}.\end{array}$$

Therefore, if

$$u(t,x)=u_i(t-i\tau,x),\quad t\in[i\tau,(i+1)\tau],\quad i\in\{0,1,2,\dots,k-1\}$$

then $u\in C(0,k\tau;X)$ and is a solution of the Cauchy problem (4) in $Q(k\tau)$.

From Theorem 2 we get an estimate on the lifespan of the solution of the Cauchy problem (1), (2).

**Corollary 1.** *Suppose that the assumptions* $(H1)$ *and* $(H2)$ *are fulfilled. Then for any number* $M>0$ *there exists* $\delta>0$ *such that the lifespan* $T^*$ *of the solution of* (1), (2) *satisfies the estimate* $T^*>M$ *provided* $|f|_X<\delta$.

Next we study the impulsive parabolic Cauchy problem (IPCP)

$$
\text{(9)} \qquad \begin{cases} \text{(a)} & \mathcal{L}_v(w(t,x)) = F(t,x,w) \qquad \text{in } Q(\infty) \setminus P \\ \text{(b)} & w(0,x) = f(x) \qquad \text{on } \mathbb{R}^n \\ \text{(c)} & w(T_k, x) = I_k(w(T_k-, x)) \quad \text{on } P_k, \quad \forall\, k \in \mathbb{N}. \end{cases}
$$

where the "impulsive source" in (9(c)) is represented by the mapping $I_k$: $X \to X$, $k \in N$.

A function $w$: $R_+ \to X$ is called a solution of the IPCP (9) if $w(t,x) \in C(\mathring{\mathbb{R}}; X)$ and satisfies (9).

The following theorem gives a condition which guarantees the existence of global solution of the IPCP (9).

**Theorem 3.** *Assume that the IPCP* (9) *satisfies the following four hypothesis:* $(H1)$, $(H2)$ *and*

$(H3)$ *The sequence of impulsive sources* $\{I_k\}$ *satisfies the estimate*

$$
\text{(10)} \qquad |I_k(f)|_X \le 2^{-m_k} |f|_X, \quad \forall\, k \in \mathbb{N}, \quad \forall\, f \in K_a = \{f \in X : |f|_X < a\}
$$

*where* $m_k = \left[\dfrac{T_{k-1} - T_k}{\tau}\right] + 1 \in N$ *and* $\tau$ *is given in Theorem 1.*

$(H4)$ *The initial condition* $f(x) \in X$ *and satisfies*

$$
|f|_X \le 2^{-m_0} a,
$$

*where* $m_0 = \left[\dfrac{T_1}{\tau}\right] + 1$.

*Then the IPCP* (9) *has a unique solution* $w(t,x) \in C(\mathring{\mathbb{R}}; X)$. *The solution* $w(t,x)$ *satisfies the estimate*

$$
|w(t,.)|_X \le a, \quad t \in \bar{R}_+.
$$

**Proof.** From Theorem 2 we get that the Cauchy problem

$$
\begin{cases} \mathcal{L}_v(u_0) = F(t,x,u_0) & \text{in } Q(m_0\tau) \\ u_0(0,x) = f(x) \end{cases}
$$

has a solution $u_0(t,x) \in C(0, m_0\tau; X)$.

Because $0 < T_1 < m_0\tau$, we define $f_1(x) = u_0(T_1, x)$.

If for some $k \in \mathbb{N}$ are defined $u_{k-1}(t,x)$ and $f_k(x)$, then we solve the Cauchy problem

$$
\begin{cases} \mathcal{L}_v(u_k) = F(t,x,u_k) & \text{in } Q(m_0\tau) \\ u_k(0,x) = I_k(f_k) & \text{on } \mathbb{R}^n \end{cases}
$$

Theorem 2 gives the existence of the solution $u_k(t,x)$ from the class $C(0, m_k\tau; X)$. Since $0 < T_{k+1} - T_k < m_k\tau$, we define $f_{k+1}(x) = u_k(T_{k+1} - T_k, x)$.

Using the above defined sequence $\{u_k(t,x)\}$ we find

$$
w(t,x) = u_k(t - T_k, x), \quad \forall\, t \in [T_k, T_{k+1}), \quad k \in \{0, 1, 2, \dots\}.
$$

The function $w(t,x)$ is a solution of (IPCP), and the uniqueness is a corollary from the uniqueness part of Theorem 1.

**Note 1.** All the theorems and proofs remain the same, if the Hölder continuity of the potential $v(x)$ is replaced with the condition of Dini. It is known that (see [4]), that this is the broadest class of potentials for which $\mathcal{L}_v(u)$ has a classical fundamental solution.

**Note 2.** The above proved results holds also in the case when the potential $v(x)$ and the initial conditions $f(x)$ belong to $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $p > 1$.

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## СЪЩЕСТВУВАНЕ НА РЕШЕНИЕ НА ЗАДАЧАТА НА КОШИ ЗА ПОЛУЛИНЕЙНОТО УРАВНЕНИЕ НА ТОПЛОПРОВОДНОСТТА


**Марин Л. Маринов**



За задачата на Коши за полулинейното параболично уравнение с потенциал е доказана оценка за времето на локално съществуване, която зависи само от максимума на потенциала и нелинейността. В случая когато решението избухва е оценено времето на съществуване на решението и са получени общи условия върху импулсите за които стандартната задача с импулси има глобално решение.