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DISCRETE MODEL FOR DEPENDENT DEFAULT TIMES

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A simple discrete time model of dependence between the default times of assets (firms) in a given portfolio is discussed. The model is based on the multivariate geometric distributions.

1. Introduction. The problem for correlation between the times to default of assets (firms) in a given portfolio is of great interest. There are two different approaches to this problem, known as structural and reduced, respectively. There are different models of both types concerning this topic (see e.g. [2], [3], [4]). In the present note we consider a simple discrete time model based on the multivariate geometric distributions. It can be classified as intensity based model. The calibration of the model is, as usual, difficult problem. For this reason the calibration is done not in the general case but in two particular cases, using the data available from the rating agencies like Standard&Poors and Moody's KMV. The model is related to the continuous time model introduced by Giesecke [1].

2. Description of the model. Let us have a portfolio consisting of $N \geq 2$ assets (firms). We make the following assumptions:

1. The time is discrete $t = 0, 1, 2, \ldots$ The calendar year consists of $T > 2$ periods (units of time).

2. The default event of an asset (firm) can be caused by different types of shocks.

In general, we assume that $M = \sum_{l=1}^{N} \binom{N}{l} = 2^N - 1$ types of shocks can occur: $\binom{N}{1} = N$ shocks specific for each firm separately; $\binom{N}{2}$ shocks specific for each pair of firms … $\binom{N}{N} = 1$ shock which causes a default event to all the firms in the portfolio.

3. The instances of shocks $\xi_1, \xi_2, \ldots, \xi_M$ are independent random variables with distributions $\Pr(\xi_j = k) = q_j^{k-1} p_j$, where $0 < p_j < 1$, $q_j = 1 - p_j$, $k = 0, 1, 2, \ldots$, $j = 1, 2, \ldots$.

Define the matrix $A = ||a_{ij}||_{N \times M}$ such that each raw corresponds to a firm, and each column corresponds to a shock. If the j–th shock causes the default event to i–th firm then $a_{ij} = 1$, and $a_{ij} = 0$ otherwise. The time to default τ_i of the i–th firm is the time when the first possible shock occurs, i.e.

$$\tau_i = \min_{a_{ij}=1, j=1,2,\ldots,N} \{\xi_1, \xi_2, \ldots, \xi_M\}, \quad i = 1, \ldots, N. \tag{1}$$

We do not assume that the firm can recover after a default event.

3. Distributions of the times to default. It is clear that the times to default are not independent. On the other hand, they have again geometric distributions.

Prorposition 1. *The distribution of* τ_i *is*

$$\Pr(\tau_i = t) = Q_i^{t-1} P_i, t = 1, 2, \ldots, \quad Q_i = \prod_{j=1}^{M} q_j^{a_{ij}}, P_i = 1 - Q_i \tag{2}$$

for $i = 1, 2, \ldots, N$. *The joint survival function (j.s.f.) of* $(\tau_1, \tau_2, \ldots, \tau_N)$ *is*

$$\begin{aligned} & s(t_1, t_2, \ldots, t_N) \\ = \quad & \Pr(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_N > t_N) = \prod_{j=1}^{M} q_j^{\max(a_{1j}t_1, \ldots, a_{Nj}t_N)}. \end{aligned} \tag{3}$$

Proof. Since $\Pr(\xi_j > k) = q_j^k, k = 1, 2, \ldots$, then by the independence of ξ_j, we calculate, taking into account that $a_{ij} = 1$ or $a_{ij} = 0$,

$$\begin{aligned} s_i(t_i) = \Pr(\tau_i > t_i) &= \prod_{a_{ij}=1, j=1,\ldots,M} \Pr(\xi_j > t_i)^{a_{ij}} \\ &= \left(\prod_{a_{ij}=1, j=1,\ldots,M} q_j^{a_{ij}} \right)^{t_i} = \left(\prod_{j=1}^{M} q_j^{a_{ij}} \right)^{t_i} =: Q_i^{t_i} \end{aligned} \tag{4}$$

for $i = 1, \ldots, N$. Therefore,

$$\Pr(\tau_i = t_i) = \Pr(\tau_i > t_i - 1) - \Pr(\tau_i > t_i) = Q_i^{t_i - 1} - Q_i^{t_i} = (1 - Q_i) Q_i^{t_i - 1}$$

for $t_i = 1, 2, \ldots$. Hence, the joint survival function, one obtains

$$\begin{aligned} & \Pr(\tau_1 > t_1, \tau_2 > t_2, \ldots, \tau_N > t_N) \\ = \; & \Pr(\min_{a_{1j}=1, j=1,\ldots,M} \{\xi_j\} > t_1, \min_{a_{2j}=1, j=1,\ldots,M} \{\xi_j\} > t_2, \ldots, \min_{a_{Nj}=1, j=1,\ldots,M} \{\xi_j\} > t_N) \\ = \; & \Pr(\xi_1 > \max_{a_{i1}=1, i=1,\ldots,N} \{t_i\}, \xi_2 > \max_{a_{i2}=1, i=1,\ldots,N} \{t_i\}, \ldots, \xi_M > \max_{a_{iM}=1, i=1,\ldots,N} \{t_i\}) \\ = \; & q_1^{\max_{a_{i1}=1, i=1,\ldots,N} \{a_i\}} q_2^{\max_{a_{i2}=1, i=1,\ldots,N} \{a_i\}} \ldots q_M^{\max_{a_{iM}=1, i=1,\ldots,N} \{t_i\}} \\ = \; & q_1^{\max\{a_{11}t_1, a_{21}t_2, \ldots, a_{N1}t_N\}} q_2^{\max\{a_{12}t_1, a_{22}t_2, \ldots, a_{N2}t_N\}} \ldots q_M^{\max\{a_{1M}t_1, a_{2M}t_2, \ldots, a_{NM}t_N\}} \\ = \; & \prod_{j=1}^{M} q_j^{\max(a_{1j}t_1, \ldots, a_{Nj}t_N)}. \end{aligned}$$

Let $s \neq r$. Setting in the last equation $t_i = 0$, for all $i \neq s, r$, we obtain the bivariate survival function of the vector (τ_s, τ_r)

$$\Pr(\tau_s > t_s, \tau_r > t_r) = \prod_{j=1}^{M} q_j^{\max(a_{sj}t_s, a_{rj}t_r)}. \tag{5}$$

Using the equations (4) and (5) we calculate the correlation coefficient between the indicators

$$\rho(I_{\{\tau_s \le T\}}, I_{\{\tau_r \le T\}}) \tag{6}$$
$$= \frac{\Pr(\tau_s > T, \tau_r > T) - \Pr(\tau_s > T)\Pr(\tau_r > T)}{\sqrt{\Pr(\tau_s > T)\Pr(\tau_r > T)(1-\Pr(\tau_s > T))(1-\Pr(\tau_r > T))}}$$
$$= \frac{\left(\prod_{j=1}^{M} q_j^{\max(a_{sj}, a_{rj})}\right)^T - Q_s^T Q_r^T}{\sqrt{Q_s^T Q_r^T (1-Q_s^T)(1-Q_r^T)}}.$$

Remark 1. 1. *The distribution of the vector* $(\tau_1, \tau_2, \ldots, \tau_N)$ *is known as* N*-variate geometric distribution.* 2. *Here we represent the characteristics of the distribution which is used in the next sections.* 3. *The general model contains* $M = 2^N - 1$ *parameters which have to be estimated. On the other hand, the information available from the Rating Agencies (e.g. Standard & Poors, Moody's KVM) is restricted to the default probability for particular period* T*, i.e.,* $\Pr\{\tau_i \le T\}$ *for every firm, and to the so called Moody's KMV default correlation, which is in fact the correlation coeefficient between the indicators* $I_{\{\tau_i \le T\}}$ *and* $I_{\{\tau_j \le T\}}$*. That's why the parameters are estimated only in two special cases of the general model.*

4. Common shock model. Consider the following case: each firm is exposed to two types of shocks. The first shock is firm specific and it does not depend on the condition of the other firms. The second one is an economy-wide shock event. Its occurrence leads to default of all firms. In this case the matrix $A = ||a_{ij}||_{N\times M}$ reduces to

$$A = \begin{bmatrix} 1 & 0 & . & . & . & 0 & 1 \\ 0 & 1 & . & . & . & 0 & 1 \\ 0 & 0 & . & . & . & 0 & 1 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 & 1 \end{bmatrix},$$

and $M = N + 1$. Denoting by $\xi_j, j = 1, 2, \ldots, N$, the instances when the specific shocks occur and by ξ – the instance when the economy-wide shock occurs, we have the following expression for τ_i:

$$\tau_i = \min(\xi_i, \xi), \quad i = 1, \ldots, N.$$

Suppose that $\xi_j \sim Ge(p_j), 0 < p_j < 1,\ q_j = 1 - p_j,\ \ j = 1, \ldots, N,\ \xi \sim Ge(p),\ 0 < p < 1,\ q = 1 - p$. Now, using the results from Section 3 (with the corresponding changes in notations), we obtain the joint survival function

$$s(t_1, t_2, \ldots, t_N) = \prod_{i=1}^{N} q_i^{t_i} q^{\max(t_1, t_2, \ldots, t_N)}.$$

This leads to the marginal survival function

$$s(t_i) = \Pr(\tau_i > t_i) = (q_i q)^{t_i}, t_i = 1, 2, \ldots;\ i = 1, 2, \ldots, N, \tag{7}$$

and bivariate survival function of the vector (τ_s, τ_r),

$$s(t_s, t_r) = q_s^{t_s} q_r^{t_r} q^{\max(t_s, t_r)}. \tag{8}$$

Using (6), (7) and (8), the correlation coefficient between the indicators $I_{\{\tau_i \le T\}}$ and

$I_{\{\tau_j \le T\}}$ takes the form

$$\rho(I_{\{\tau_i \le T\}}, I_{\{\tau_j \le T\}}) = \frac{(q_i q_j q)^T - (q_i q)^T (q_j q)^T}{\sqrt{(q_i q)^T (q_j q)^T (1-(q_i q)^T)(1-(q_j q)^T)}}. \tag{9}$$

5. Model Calibration. It was mentioned above that the calibration of the model is a difficult problem because of the information that is available. We calibrate the model using the Moody's KMV probabilities $\hat{p}_i, i = 1, \ldots, N$, for one year default and the so called Moody's KMV default correlation $\hat{\rho}_{ij}, i = 1, \ldots, N-1,\ j = i+1, \ldots, N$, which is in fact the correlation coefficient between the indicators. It was assumed earlier that the year is divided into T periods. Using the available data and equations (7), and (9), the following system is obtained:

$$\hat{p}_i = P(\tau_i \le T) = 1 - (q q_i)^T,$$

$i = 1, 2, \ldots, N$,

$$\hat{\rho}_{i,j} = \rho(I_{\{\tau_i \le T\}}, I_{\{\tau_j \le T\}}) = \frac{(q_i q_j q)^T - (q_i q)^T (q_j q)^T}{\sqrt{(q_i q)^T (q_j q)^T (1-(q_i q)^T)(1-(q_j q)^T)}},$$

$i = 1, 2, \ldots, N-1, j = i+1, \ldots, N$. The system contains $N(N+1)/2$ equations for $N+1$ unknown parameters. So, q should satisfy $N(N-1)/2$ equations as follows:

$$q = \left(\frac{(1-\hat{p}_i)(1-\hat{p}_j)}{\hat{\rho}_{i,j}\sqrt{\hat{p}_i(1-\hat{p}_i)\hat{p}_j(1-\hat{p}_j)} + (1-\hat{p}_i)(1-\hat{p}_j)} \right)^{\frac{1}{T}},$$

$i = 1, 2, \ldots, N-1, j = i+1, \ldots, N$. Evidently, it is not possible to obtain the exact solution of the system. The least square method gives the following approximate solution

$$q = \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(\frac{(1-\hat{p}_i)(1-\hat{p}_j)}{\hat{\rho}_{i,j}\sqrt{\hat{p}_i(1-\hat{p}_i)\hat{p}_j(1-\hat{p}_j)} + (1-\hat{p}_i)(1-\hat{p}_j)} \right)^{\frac{1}{T}}.$$

Then

$$q_i = (1-\hat{p}_i)^{\frac{1}{T}}\, q^{-1},\ i = 1, 2, \ldots, N.$$

6. Pair default model. Another particular model, which better corresponds to the data available from rating agencies, is obtained from the general model if we assume that there are N shocks specific for each firm and $N(N-1)/2$ shocks which cause default of each pair of firms. For the sake of convenience we will denote the corresponding random variables by $\xi_{ij} \sim Ge(p_{ij}),\ 0 < p_{ij} < 1, q_{ij} = 1 - p_{ij}, i = 1, 2, \ldots, N, j = i, i+1, \ldots, N$. In this way the r.v. $\xi_{ii},\ i = 1, 2, \ldots, N$ is the instance when the shock, specific for the i–th firm, occurs, while the r.v. $\xi_{ij}, i < j$ is the instance when the shock leading to the simultaneous default of the i–th and j–th firms occurs.

The time of default of the ith firm is defined by

$$\tau_i = \min\{\xi_{1i}, \xi_{2i}, \ldots, \xi_{ii}, \xi_{i,i+1}, \ldots, \xi_{iN}\},$$

for $i = 1, 2, \ldots, N$.

The marginal survival functions are immediately calculated as follows (using the independence of ξs):

$$\Pr(\tau_i > t_i) = \Pr(\min\{\xi_{1i}, \xi_{2i}, \ldots, \xi_{ii}, \xi_{i,i+1}, \ldots, \xi_{iN}\} > t_i) \tag{10}$$
$$= \Pr(\xi_{1i} > t_i, \xi_{2i} > t_i, \ldots, \xi_{ii} > t_i, \xi_{i,i+1} > t_i, \ldots, \xi_{iN} > t_i)$$
$$= \Pr(\xi_{1i} > t_i) \Pr(\xi_{2i} > t_i) \ldots \Pr(\xi_{ii} > t_i) \Pr(\xi_{i,i+1} > t_i) \ldots \Pr(\xi_{iN} > t_i)$$
$$= \left(\prod_{s=1}^{i} q_{si} \prod_{l=i+1}^{N} q_{il} \right)^{t_i}.$$

Thus, $\Pr(\tau_i = t_i) = \left(\prod_{s=1}^{i} q_{si} \prod_{j=i+1}^{N} q_{ij} \right)^{t_i - 1} \left(1 - \prod_{s=1}^{i} q_{si} \prod_{j=i+1}^{N} q_{ij} \right)$, i.e. the random variable $\tau_i \sim Ge(P_i), i = 1, 2, \ldots, N$, where $P_i = 1 - \prod_{s=1}^{i} q_{si} \prod_{j=i+1}^{N} q_{ij}$, $Q_i = \prod_{s=1}^{i} q_{si} \prod_{j=i+1}^{N} q_{ij}$.

Using the results from Section 3 (with corresponding change of notations), one obtains for the joint survival function

$$s(t_1, t_2, \ldots, t_N) = \Pr(\tau_1 > t_1, \ldots, \tau_s > t_s, \ldots, \tau_r > t_r, \ldots, \tau_N > t_N) \tag{11}$$
$$= \left(\prod_{i=1}^{N} q_{ii}^{t_i} \right) \left(\prod_{i=1}^{N-1} \prod_{j=i+1}^{N} q_{ij}^{\max\{t_i, t_j\}} \right).$$

Setting in (11) $t_i = 0$ for $i \neq s, r$, we obtain the bivariate survival function of the vector (τ_s, τ_r), $1 \leq s < r \leq N$,

$$\Pr(\tau_s > t_s, \tau_r > t_r) \tag{12}$$
$$= q_{sr}^{\max\{t_s, t_r\}} \left(\prod_{i=1}^{s} q_i s \prod_{j=s+1, j \neq r}^{N} q_{sj} \right)^{t_s} \left(\prod_{i=1, i \neq s}^{r} q_i r \prod_{j=r+1}^{N} q_{rj} \right)^{t_r}$$
$$= q_{sr}^{\max\{t_s, t_r\} - t_s - t_r} \left(\prod_{i=1}^{s} q_i s \prod_{j=s+1}^{N} q_{sj} \right)^{t_s} \left(\prod_{i=1}^{r} q_i r \prod_{j=r+1}^{N} q_{rj} \right)^{t_r}$$
$$= q_{sr}^{-\min\{t_s, t_r\}} Q_s^{t_s} Q_r^{T_r}.$$

The last equation, (10), and (6) give the correlation coefficient between indicators

$$\rho(I_{\{\tau_s \leq T\}}, I_{\{\tau_r \leq T\}}) \tag{13}$$
$$= \frac{q_{sr}^{-T} Q_s^T Q_r^T - Q_s^T Q_r^T}{\sqrt{Q_s^T Q_r^T (1 - Q_s^T)(1 - Q_r^T)}} = \frac{1 - q_{sr}^T}{q_{sr}^T} \sqrt{\frac{Q_s^T Q_r^T}{(1 - Q_s^T)(1 - Q_r^T)}}$$

for $s = 1, 2, \ldots, N - 1$, $r = s + 1, \ldots, N$.

7. Model Calibration. The calibration of the model is based again on the probabilities for one year default $\hat{p}_s = P(\tau_s \leq T)$, $s = 1, \ldots, N$ and the Moody's KMV correlation between indicators $\hat{\rho}_{sr} = \rho(I_{\{\tau_s \leq T\}}, I_{\{\tau_r \leq T\}})$, $s = 1, \ldots, N - 1$; $r = s + 1, \ldots, N$. Using

(10) and (13), we obtain the following system of equations

$$\hat{p}_s = P(\tau_s \le T) = 1 - Q_s^T = 1 - \left(\prod_{i=1}^{s} q_{is} \prod_{j=s+1}^{N} q_{sj}\right)^T$$

for $s = 1, 2, \dots, N$, and

$$\hat{\rho}_{sr} = \frac{q_{sr}^{-T} Q_s^T Q_r^T - Q_s^T Q_r^T}{\sqrt{Q_s^T Q_r^T (1 - Q_s^T)(1 - Q_r^T)}} = \frac{1 - q_{sr}^T}{q_{sr}^T} \sqrt{\frac{Q_s^T Q_r^T}{(1 - Q_s^T)(1 - Q_r^T)}}.$$

for $s = 1, 2, \dots, N-1; r = s+1, \dots, N$.

The system contains $N(N+1)/2$ equations for $N(N+1)/2$ unknown parameters. Its unique solution is given by

$$q_{sr} = \left(\frac{\sqrt{(1-\hat{p}_s)(1-\hat{p}_r)}}{\hat{\rho}_{sr}\sqrt{\hat{p}_s\hat{p}_r} + \sqrt{(1-\hat{p}_s)(1-\hat{p}_r)}}\right)^{1/T}$$

for $s = 1, 2, \dots, N-1$, $r = s+1, \dots, N$ and

$$q_{ss} = (1 - \hat{p}_s)^{1/T} / \left(\prod_{i=1}^{s-1} q_{is} \prod_{j=s+1}^{N} q_{sj}\right)$$

for $s = 1, 2, \dots, N$.

8. Conclusion remarks. An application of the model to the estimation of expected losses in CDO tranches will be published later. This work is partially supported by NFSI grant No.VU-MI-105/2005.

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ДИСКРЕТЕН МОДЕЛ НА ЗАВИСИМИ ВРЕМЕНА ЗА ФАЛИТ

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Разглежда се дискретен модел на зависими времена за фалит на фирми, съставляващи даден портфейл. Моделът е основан на многомерно геометрично разпределение.