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ON THE MEASURABILITY OF SETS OF SPHERES IN THE GALILEAN SPACE*

Adrijan V. Borisov

The measurable sets of spheres and the corresponding invariant densities with respect to the similarity group and some its subgroups are described.

1. Introduction. The Galilean space G_3 is defined as a 3-dimensional projective space $P_3(R)$ in which the absolute consists of a real plane ω and a real line f in ω together with an elliptic involution ε of the points of f. All regular projectivities commuting with ε and transforming the absolute figure $\{\omega, f\}$ into itself, form the 8-parametric group H_8 of similarities of G_3. In affine coordinates H_8 has the form

$$\begin{aligned} x' &= a_{11} + a_{12}x, \\ y' &= a_{21} + a_{22}x + a_{23}y\cos\varphi + a_{23}z\sin\varphi, \\ z' &= a_{31} + a_{32}x - a_{23}y\sin\varphi + a_{23}z\cos\varphi, \end{aligned} \tag{1}$$

where a_{ij} and φ are real numbers and $a_{12} > 0$, $a_{23} > 0$.

In particular, if:

(i) $a_{12} = a_{23} = \alpha$, then the subgroup $H_7 \subset H_8$ consists of transformations which map line segments into proportional segments with coefficient of proportionality α and preserving angles between planes and lines respectively. The group H_7 is called the group of **equiform transformations** of the Galilean space G_3 [4].

(ii) $a_{12} = a_{23} = 1$, then we have the subgroup $B_6 \subset H_8$ - the group of **Galilean motions** or the **group of isometries** of the Galilean space G_3.

The differential geometry of the Galilean space G_3 has been largely developed by O. Röschel in [5].

The aim of this paper is to study the measurability in the sense of M.I.Stoka [6] and G.I.Drinfeld [2] of sets of spheres with respect to H_8 and the indicated above subgroups.

2. Measurability with respect to H_8. Quadrics in G_3 have been treated in [7] but a more complete metric classification of regular and singular quadrics has been given in [3].

Let a quadric Σ in the space G_3 be given by equation of the form

$$x^2 + Bx - 2Cy - 2Dz + E = 0, \tag{2}$$

where B, C, D, E are real parameters and $C^2 + D^2 \neq 0$.

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Remark 2.1. *We note that in* [3] *and in* [5] *the quadric* Σ *is called the isotropic circular cylinder and the* **point sphere** (*Punktkugel*) *of the radius*

$$R = \sqrt{C^2 + D^2}, \tag{3}$$

respectively. We call this quadric the sphere of the Galilean space G_3.

Under the action of (1) the sphere $\Sigma(B, C, D, E)$ is transformed into the sphere $\bar{\Sigma}(\bar{B}, \bar{C}, \bar{D}, \bar{E})$:

$$\begin{aligned}
\bar{B} &= -2a_{11} + a_{12}B + 2\frac{a_{12}}{a_{23}}[(a_{22}\cos\varphi - a_{32}\sin\varphi)C + (a_{22}\sin\varphi + a_{32}\cos\varphi)D] \\
\bar{C} &= \frac{a_{12}^2}{a_{23}}(C\cos\varphi + D\sin\varphi), \\
\bar{D} &= \frac{a_{12}^2}{a_{23}}(-C\sin\varphi + D\cos\varphi), \\
\bar{E} &= a_{11}^2 - a_{11}a_{12}B + 2\frac{a_{12}^2}{a_{23}}\left[\left(\frac{a_{32}\sin\varphi - a_{22}\cos\varphi}{a_{12}}a_{11} + a_{21}\cos\varphi - a_{31}\sin\varphi\right)C + \right. \\
&\left.\left(-\frac{a_{32}\cos\varphi + a_{22}\sin\varphi}{a_{12}}a_{11} + a_{21}\sin\varphi + a_{31}\cos\varphi\right)D\right] + a_{12}^2E.
\end{aligned} \tag{4}$$

The transformations (4) form the so-called associated group $\bar{H}_8$ of H_8 [6;p.34]. $\bar{H}_8$ is isomorphic to H_8 and the invariant density with respect to H_8 of the spheres (2), if it exists, coincides with the invariant density with respect to $\bar{H}_8$ of the points (B, C, D, E) in the set of parameters [6;p.33]. The infinitesimal operators of $\bar{H}_8$ are

$$Y_1 = 2\frac{\partial}{\partial B} + B\frac{\partial}{\partial E}, \quad Y_2 = C\frac{\partial}{\partial E}, \; Y_3 = C\frac{\partial}{\partial B},$$

$$Y_4 = D\frac{\partial}{\partial E}, \quad Y_5 = D\frac{\partial}{\partial B}, \quad Y_6 = D\frac{\partial}{\partial C} - C\frac{\partial}{\partial D},$$

$$Y_7 = C\frac{\partial}{\partial C} + D\frac{\partial}{\partial D}, \quad Y_8 = B\frac{\partial}{\partial B} + 2C\frac{\partial}{\partial C} + 2D\frac{\partial}{\partial D} + 2E\frac{\partial}{\partial E}.$$

Obviously, Y_2, Y_3, Y_6 and Y_7 are arcwise unconnected and

$$Y_8 = 2\frac{E}{C}Y_2 + \frac{B}{C}Y_3 + 2Y_7.$$

Since

$$Y_2\left(2\frac{E}{C}\right) + Y_3\left(\frac{B}{C}\right) + Y_7(2) \neq 0,$$

it follows immediately:

Theorem 2.2. *The set* (2) *of sphere* Σ *in* G_3 *is not measurable under similarity group* H_8 *and it has not measurable subgroups.*

3. Measurability with respect to the subgroup H_7**.** The associated group $\bar{H}_7$ of the group H_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ and $Z_7 = B\frac{\partial}{\partial B} + C\frac{\partial}{\partial C} + D\frac{\partial}{\partial D} + 2E\frac{\partial}{\partial E}$. The infinitesimal operators Y_2, Y_3, Y_6 and Z_7 are arcwise unconnected and

$$Y_1 = \frac{B}{C}Y_2 + \frac{2}{C}Y_3, \quad Y_4 = \frac{D}{C}Y_2, \quad Y_5 = \frac{D}{C}Y_3.$$

Since

$$Y_2\left(\frac{B}{C}\right) + Y_3\left(\frac{2}{C}\right) = 0, \quad Y_2\left(\frac{D}{C}\right) = 0, \quad Y_3\left(\frac{D}{C}\right) = 0, \tag{5}$$

the corresponding associated group $\bar{H}_7$ is measurable and the integral invariant function $f = f(B, C, D, E)$ satisfies the system of R. Deltheil [1;p.28], [6;p.11]

$$Y_2(f) = 0, \quad Y_3(f) = 0, \quad Y_6(f) = 0, \quad Z_7(f) + 5f = 0. \tag{6}$$

The system (6) has the solution

$$f = \frac{h}{(C^2 + D^2)^{\frac{5}{2}}},$$

where $h =$ const. Thus, we can state:

Theorem 3.1. *The set* (2) *of spheres* Σ *in* G_3 *is measurable with respect to the group of equiform transformations* H_7 *and has the invariant density*

$$d\Sigma = \frac{1}{(C^2 + D^2)^{\frac{5}{2}}} dB \wedge dC \wedge dD \wedge dE. \tag{7}$$

By (3) the formula (7) can be written as

$$d\Sigma = \frac{1}{R^5} dB \wedge dC \wedge dD \wedge dE. \tag{8}$$

Differentiating (3), we have

$$dR = \frac{1}{R}(CdC + DdD). \tag{9}$$

By exterior product of (9) and $dB \wedge dD \wedge dE$ and $dB \wedge dC \wedge dE$, we obtain

$$dB \wedge dC \wedge dD \wedge dE = -\frac{R}{C} dR \wedge dB \wedge dD \wedge dE,$$

$$dB \wedge dC \wedge dD \wedge dE = \frac{R}{D} dR \wedge dB \wedge dC \wedge dE,$$

respectively. Thus, the formula of density (7) becomes

$$d\Sigma = \frac{1}{CR^4} dR \wedge dB \wedge dD \wedge dE, \tag{10}$$

$$d\Sigma = \frac{1}{DR^4} dR \wedge dB \wedge dC \wedge dE.$$

We summarize the foregoing results in the following

Corollary 3.2. *The density of the spheres* (2) *in* G_3 *satisfies the relations* (8) *and* (10).

4. Measurability with respect to the subgroup B_6. The corresponding associated group $\bar{B}_6$ of the group B_6 has infinitesimal operators $Y_1 = \frac{B}{C}Y_2 + \frac{2}{C}Y_3, Y_2, Y_3, Y_4 = \frac{D}{C}Y_2, Y_5 = \frac{D}{C}Y_3, Y_6$ and, consequently, acts intransitively on the set of spheres (2), i.e. the set of spheres (2) is not measurable with respect to the group B_6. From (5) and

$Y_2(f) = 0,\ Y_3(f) = 0,\ Y_6(f) = 0$ we deduce that the set (2) has the measurable subset

$$C^2 + D^2 = R^2, \tag{11}$$

where $R = \mathrm{const}$, i.e. the spheres (2) have a constant radius.

Theorem 4.1. *The set of the spheres* (2) *is not measurable with respect to the isometry group* B_6. *It has the measurable subset* (11) *with the invariant density*

$$d\Sigma = dB \wedge dC \wedge dE = dB \wedge dD \wedge dE.$$

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South-West University "Neofit Rilski"
Faculty of Mathematics and Natural Science
Department of Mathematics
66, Ivan Mihailov Str.
2700 Blagoevgrad, Bulgaria
e-mail: adribor@aix.swu.bg

ВЪРХУ ИЗМЕРИМОСТТА НА МНОЖЕСТВА ОТ СФЕРИ В ГАЛИЛЕЕВО ПРОСТРАНСТВО

Адриян Върбанов Борисов

Описани са измеримите множества от сфери в галилеево пространство и са намерени съответните им инвариантни гъстоти относно групата на подобностите и някои нейни подгрупи.