МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2007
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2007
*Proceedings of the Thirty Sixth Spring Conference of
the Union of Bulgarian Mathematicians
St. Konstantin & Elena resort, Varna, April 2–6, 2007*

LOCAL VARIATION METHOD TO CALCULATE TWO DIMENSIONAL GRAVITY WAVES IN CONTAINER*

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We study the dynamics of two-dimensional gravity waves of an inviscid incompressible fluid in a channel. The fluid flow is assumed to be irrotational. We have energy dissipation along the triple contact line. A numerical code is developed to study the dependence of the time evolution on the free line. We use a standard numerical scheme to solve the interfacial flow problems. The main element of this scheme is calculation of the harmonic potential. For this we develop a modified numerical method of the local variations method to solve the Laplace equation on non-rectangular domain with mixed boundary conditions. We show that with the suggested numerical algorithm the behavior of the fluid in a container can be effectively studied.

1. Introduction. It is of fundamental and engineering importance to investigate liquid sloshing in a partially filled container for unspecified motion that is a complex theoretical problem. Numerous studies are devoted to investigate gravity waves in confined region using different approaches and techniques. The asymptotic methods for determining the frequencies of standing gravity waves are well known, at least for vertical boundaries. In [1] a weakly-nonlinear theory is formulated for standing waves in infinite depth and in [2] – case of arbitrary depth. To calculate the dynamics of an incompressible fluid with free surface different numerical methods are used [3–6]. Most of the research on gravity waves on a free liquid surface is concentrated on waves propagation. The way how dissipative processes on the triple contact line influences the gravity waves is not enough investigated. Usually only not completely realistic extreme cases when the free surface meets the boundary orthogonally or the contact line remains fixed throughout the motion are considered. More realistic assumption is that dissipation is large but finite. One of the basic models [7] uses the idea that the fluid particle on the triple contact line can move, but when it moves a friction force, proportional to the point velocity, acts on it which impedes its motion. Until now, for this model there isn't any solution available for dynamic behavior of points on the triple contact line. When friction force is added to the model, then the dynamic equation becomes much more complicated. For this case in each time step Laplace equation must be solved for more complex domain with more complicated boundary conditions. In this work we would like to propose effective numerical method to solve this problem. To this end we modify the

*2000 Mathematics Subject Classification: 35J50, 65K10, 76B15
Key words: Laplace equation, Gravity waves, Local variation method

local variation method. This method is effective to solve partial differential equation with mixed boundary conditions in a variety of domains. Using this method we obtain for the first time dynamic solution for the motion of the fluid when the described friction force is taken into account. Till now only spontaneous quasi-static relaxation dynamics of liquid on solid surfaces in the partial wetting regime was studied numerically [8], [9].

2. Problem formulation. We consider motion of an inviscid and incompressible fluid inside 2D rectangular container under the action of the gravity $\mathbf{g}$ and the friction force $\mathbf{F} = -\gamma^* \mathbf{v}^*$ (γ^* can be considered as a friction coefficient) (see Fig. 1).

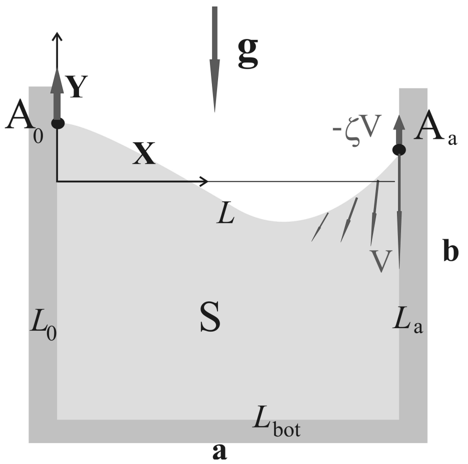

Fig. 1. Definition sketch

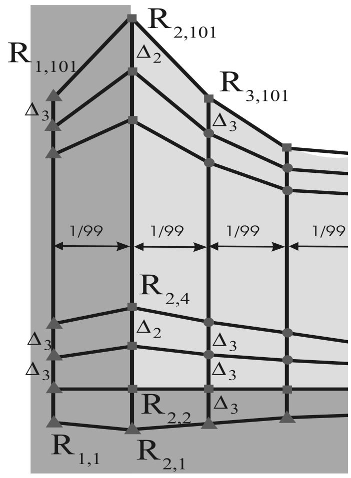

Fig. 2. Schematic graphic of a part of the grid lines and grid points in solution domain

The distance between the walls L_0 and L_a is a. L_{bot} is the tank bottom, the depth is b. The fluid flow is assumed to be irrotational, therefore it can be described in terms of a velocity potential φ ($\mathbf{v} = \operatorname{grad}\varphi$). We use a Cartesian coordinate system (x/a, y/a). The x-axis coincides with the non perturbation position of the free surface and the y-axis is directed vertically upward. We choose the channel length a as a characteristic length, and $\sqrt{a/g}$ as a characteristic time. We use dimensionless variables x, y, t, $\mathbf{v}$ and the renormalized potential φ defined as follows: $t = t^* \sqrt{g/a}$, $\gamma = \gamma^* \sqrt{a/g}$, $\varphi = \varphi^*/a\sqrt{ag}$, where g is the gravity acceleration. Potential φ must satisfy Laplace's equation in S:

$$\nabla^2 \varphi\,(R,t) = 0, \quad R(x,y) \in \mathrm{S}. \tag{1}$$

The container bottom and walls are rigid and impermeable, therefore

$$\partial\varphi(R)/\partial x = 0, \quad R \in \{L_0, L_a\}, \quad \partial\varphi(R)/\partial y = 0, \quad R \in L_{bot}. \tag{2}$$

The dynamic boundary condition on the free line L is based on the Bernoulli equation and is given at the initial points by

$$\partial\varphi(R)/\partial t = -v(R)^2/2 - R_y, \quad R(x,y) \in L/\{A_0, A_a\}, \tag{3}$$

and at the boundary points A_0, A_a of L by

$$\partial\varphi(R)/\partial t = -v(R)^2/2 - R_y - \gamma\varphi, \quad R \in \{A_0, A_a\}. \tag{4}$$

3. Numerical method. The motion of the free line $L(t)$ is simulated by performance of the following steps: 1. Specify the initial free line $L(0)$ and the distribution of the harmonic potential $\varphi(0)$ over $L(0)$; 2. Solve the equation (2) subject to the conditions (3) and $\varphi(R) = \varphi(R, 0)$, $R \in L(0)$; 3. Compute the velocity at the free line; 4. Update the position of the free line; 5. Update the harmonic potential at the free line using equations (3), (4) (for details see [10]).

Solving Laplace equation. The principle problem is to solve (1) with boundary conditions (2) and

$$\varphi(R) = \varphi(L, t), \ R \in L. \tag{5}$$

To solve this problem we obtain the minimum $f_{\min}$ of the functional

$$F(f) = \iint_V \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy \tag{6}$$

in set of functions f which satisfy the boundary conditions

$$\partial f(R)/\partial x = 0, \quad R \in \{L_0, L_a\}, \quad \partial f(R)/\partial y = 0, \quad R \in L_{bot}, \tag{7}$$

$$f(R) = \varphi(L, t), R \in L(t). \tag{8}$$

The function $f_{\min}$ satisfies the conditions (1), (2), (5). We obtain this function numerically.

We cover the domain S by a two-dimensional grid of lines, as it is shown in fig 2. To satisfy the boundary conditions (7) we add three grid lines outside of S. Denote by $R_{i,j} = (x_{i,j}, y_{i,j})$, $i = 1, \dots, 101$, $j = 1, \dots, 101$ the grid points. We have:

$$|x_{i+1,j} - x_{i,j}| = 1/99, i = 1, \dots, 100,$$
$$|y_{i,j} - y_{i,j+1}| = \Delta_i = |y_{i,2} - y_{i,101}|/99, \quad i = 1, \dots, 101; \ j = 1, \dots, 100.$$

On the grid points we approximate f by $F_{i,j}$. The condition (8) is approximated by

$$F_{i,101} = \varphi(R_{i,101}). \tag{9}$$

The condition (7) is approximated by the relations

$$\begin{aligned} &F_{1,j} = F_{3,j}, \ F_{101,j} = F_{99,j}, \ j = 2, 100; \\ &F_{i,1} = F_{i,3}, \ i = 2, 100; \\ &F_{1,1} = F_{3,3}, \ F_{1,101} = F_{3,99}. \end{aligned} \tag{10}$$

We obtain the minimum of the function of the variables $F_{i,j}$, $i = 2, 100$; $j = 2, 100$

$$F(F_{i.j}) = \sum_{i,j}^{\substack{i=2,100\\ j=2,100}} \left(\left[D_x^{i,j}(F_{i.j})\right]^2 + \left[D_y^{i,j}(F_{i.j})\right]^2\right) S_{i,j}; \tag{11}$$

where $S_{i,j}$ is the area of the trapezoid $R_{i,j}$, $R_{i+1,j}$, $R_{i+1,j+1}$, $R_{i,j+1}$;

$$D_y^{i,j} = \frac{F_{i,j+1} - F_{i,j}}{2\Delta_i} + \frac{F_{i+1,j+1} - F_{i+1,j}}{2\Delta_{i+1}}$$

$$D_x^{i,j} = \frac{1}{2\cos\theta_j} \frac{F_{i+1,j} - F_{i,j}}{\|F_{i+1,j} - F_{i,j}\|} - \frac{(\operatorname{tg}\theta_j + \operatorname{tg}\theta_{j+1})}{2} D_y^{i,j} + \frac{1}{2\cos\theta_{j+1}} \frac{F_{i+1,j+1} - F_{i,j+1}}{\|F_{i+1,j+1} - F_{i,j+1}\|}$$

with boundary conditions (9), (10). The condition (9) is fixed, but the condition (10) is not given initially. We overcome this difficulty using Local variations method we've modified. This method is based on an iterative approach. When we change a value of the function (11) at inner grid point $R_{x,3}$, $R_{x,99}$ or $R_{3,y}$, then we change the value at corresponding boundary point using condition (10).

Finding the position of free line at consecutive moments of time. At the time moment k we find φ at points $R_{i,j}$ of the grid and after that we calculate the velocities at the points $\mathrm{R}_{i,101}$ that belong to the free line $L\,(k)$. After that we evaluate the displacements of the positions of the points $R_{i,101}$ for time step Δt=0.02. This is how we find the position of these points at the next time moment. From Eqs. (4), (5) we find the values of φ at these points. By cubic spline approximations we determine all the points of the free line $L\,(k+1)$ and $\varphi\,(L\,(k+1))$.

4. Numerical results. The independent parameters are: the initial free line $L\,(0)$, the distribution of the harmonic potential $\varphi\,(0)$ over $L\,(0)$, a/b, γ. All results given here are for $a/b = 1$.

4.1 Without friction. To test numerical method we consider first the case in which the friction force is zero. We take the same initial condition as in the obtained asymptotic solution [1] for the full nonlinear problem for $\varepsilon = 0.157$. We can compare our results with asymptotic solution because the wave amplitude is small and the distance between L and the container bottom is large (see Fig. 3). We find good correspondence between the two results. In Figure 4 numerically obtained harmonic potential φ in time step 25 is shown. At each time step, when we calculate φ, we analyze how the function obtained by the minimizing process approximates directly the Laplace equation. For this in each inner grid point we calculate $\varepsilon_{i,j}(t) = \left|\nabla^2\varphi\,(t, R_{i,j})\right|$ by the second-order centered-difference approximation of $\nabla^2\varphi\,(t, R_{i,j})$. We obtain that for all time moments $\max\,(\varepsilon_{i,j} \in E) = 0.009$, $\mathrm{mean}\,(\varepsilon_{i,j} \in E) = 0.6\ 10^{-6}$, $E \equiv \{\varepsilon_{i,j}\,(t)\,, i = 2, \ldots, 99,$ $j = 2, \ldots, 99,\ t = 1, \ldots, 127\}$

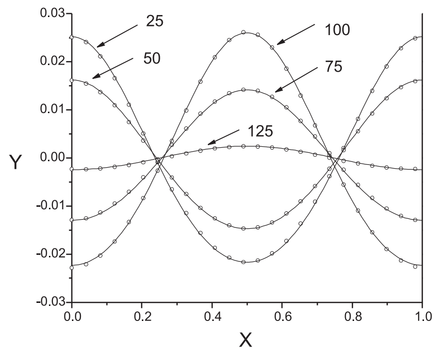

Fig. 3. With circles is shown the free line, obtained with our calculations at different moments of time. With solid line nonlinear solution obtained in [1] is shown

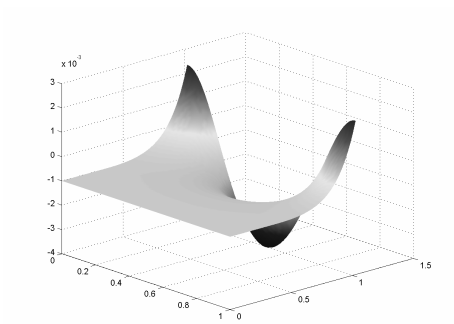

Fig. 4 The harmonic potential φ in time step T=25

In asymptotic solutions in this case the free line is approaching the solid surface at an angle of 90° at any moment of time. Numerical results correspond to this boundary condition. We obtain contact angle in each time step by linear approximation. Mean contact angle is 90.01°, the standard deviation is 0.26°, max derivation is 0.5°.

4.2 With friction. In this case we have the same initial conditions as in the case 4.1, but now the friction force $\gamma = 0.2$ is added at the boundary points of the free line. In Figure 5 we compare cases 4.1 and 4.2. For this case we calculate again how the Laplace equation is obtained in inner grid points with second-order centered-difference approximation. We obtain that for all moments of time with first cycle good approximation $\max \varepsilon_{i,j} = 0.01$, mean $\varepsilon_{i,j} = 10^{-6}$. In this case the fluid free line is approaching the solid surface at a contact angle, which changes with the time as it shown in Fig. 5.

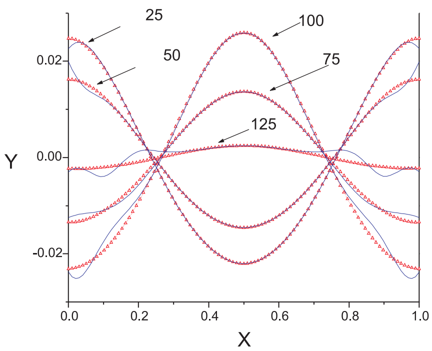

Fig. 5. The position of the line L at moments of time $T = \{25; 50; 75; 100; 125\}$ in the two cases. The position of the line L in the case 4.2 is shown with solid line. The position of the line L in the case 4.1 is shown with triangles.

The obtained results show that with the suggested numerical algorithm we can effectively solve the Laplace equation with mixed boundary condition in non-orthogonal domain and we can study the behavior of the fluid in a vessel when friction forces at the contact line are present.

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МЕТОД НА ЛОКАЛНИТЕ ВАРИАЦИИ ЗА ПОЛУЧАВАНЕ НА ДВУМЕРНИ ГРАВИТАЦИОННИ ВЪЛНИ В КОНТЕЙНЕР

Димитър С. Илиев

Изследваме динамиката на двумерни гравитационни вълни на идеален несвиваем флуид в канал. Флуидното поле се предполага потенциално. Разглеждаме модел в който има дисипация на енергията на трифазната контактна линия. Разработен е числен алгоритъм за изследване на зависимостта от времето на еволюцията на свободната линия. Използваме стандартна числена схема за решаване задачата за движение на флуид със свободна граница. Основен елемент на тази схема е намирането на потенциала. За решаването на този проблем развиваме модификация на числения метод на локалните вариации за решаване на уравнението на Лаплас за неортогонална област и смесени гранични условия. Показваме че с предлагания числен алгоритъм може ефективно да се изследва поведението на флуид в контейнер.