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DISPLACEMENTS OF ELASTIC LINK FROM OPEN-LOOP KINEMATIC CHAIN (MANIPULATOR)*

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In this paper an approach for defining the displacements of an elastic link from an open-loop kinematic chain (manipulator) is presented, supposing that the displacements are small. The approach is based on the differential equation of an elastic line of a bent beam and the D'Alembert principle. The law of motion and the equation of the elastic line are considered and solved together. The cases of the simplest motions of an elastic link with a fixed point mass at its end are analyzed. The general case of motion of this link when it appears to be the last one in the structure of manipulating system is also analyzed. An example of two-link $2R$ plane manipulator with second flexible link is given.

1. Introduction. One of the most important problems in Robotics is the effective real time obtaining of the dynamic equations of motion for the real-time simulation and control purposes, i.e. as well as for effective investigations and manipulator design, so for realizing effective control algorithms. After geometrical description of the concrete structure, the next step is the kinematical modelling which may be realized on the base of the different parameterizations of the rotation group and the different algebraic descriptions, using homogeneous matrices and rotation matrices. Further on, the dynamical modeling of manipulators can be realized on the base on the Lagrange's equations, Newton-Euler recursive equations, the equations of D'Alembert, Gauss, Appel, Kane, etc. We will not referee here the so many papers and books on this subject. The robot manipulators are divided in some groups: rigid body manipulators ([1], [2], [5], [3], [4], etc.), flexible links manipulators ([11], [13], [12], [14], [7], [15], [8], etc.), manipulators with flexible joints ([9], [6], etc.), and their different combinations. This is according to the structure and the mechanical models. And what about according to the computations. In the recent years symbolic computations and parallel algorithms are widely used for efficient modeling and computations. On the one hand the symbolic computations give the possibilities for analytical evaluation, discussions and corrections, on the other hand the parallel algorithms reduce the computational time which is quite important for on-line simulations and control. Having in mind this study, namely modeling of flexible link manipulators, the main algorithms are based on the following models: the finite element method, the Ritz method and using Lagrange, Newton-Euler and Hamilton equations of motions. In the present work, an approach for defining the displacements of

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an elastic link in open-loop kinematical chain (manipulator) is presented, supposing that the displacements are small. It is based on the differential equation of an elastic line of a bent beam and the D'Alembert principle. An example of two-link plain manipulator, where the second link is flexible, is given.

2. Problem Statement. Let us consider an example of a cantilever with a fixed point mass at its end (Fig. 1). If the beam stiffness is high enough, then it could be supposed that the beam is massless, and the only force which loads, it is the gravity force of a point mass. We accept that the beam stiffness is big enough if the beam does not undergo deformation under the influence only of its own mass. If the displacements are small, then the elastic line of the beam are defined as follows

$$EI_z \frac{d^2y}{dx^2} = -M_z, \quad M_z = mg(l - x), \tag{1}$$

where M_z is the bending moment, l is the length of the beam, I_z is the inertia moment of the beam section, m is the mass of the point mass, and g is the gravitational constant.

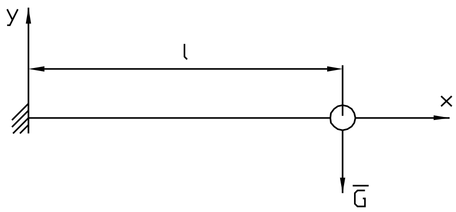

Fig. 1. Cantilever with a fixed point mass at its end

If the beam moves around some fixed point, then according to the principle of D'Alembert an additional inertia force $\overline{\Phi}$ is added. This force is oriented out from the point mass trajectory, it is equal to the product of the mass and its accelerations, and it exists during the whole motion. If the movement is given or known in advance, then the inertia force can be found in every moment. The displacements of the beam can also be obtained.

The main idea of this algorithm is: 1. to find acceleration of the point mass supposing that the link is undeformable (rigid). This idea is based on the assumption that the elastic displacements are small; 2. to compose an expression of the inertia force of the point mass; 3. to compose a differential equation of the bent elastic line of flexible link; 4. to find the elastic displacements. Further on, the simplest cases of movement of a massless beam with a fixed point mass are briefly analyzed.

3. Movement of a Massless Beam with a Fixed Point. We consider the following four cases:

CASE 1. Rotation of a Flexible Link in Vertical Plane with Constant Velocity.

Let us consider Fig. 2. A flexible link with fixed point mass rotates around Oz axis in the vertical plane. The mass of the link is neglected and the elastic displacements of the link are accepted to be small. $\overline{\Phi}$ does not generate a moment since there is no vertical component in the movable coordinate system $Ox'y'z'$, i.e.

$$\Phi_{y'} = 0. \tag{2}$$

The only force which generates a moment with respect to the point of hanging O is the gravity force of the point mass. Its vertical components are

$$G_{y'} = G\cos\theta = mg\cos\theta. \tag{3}$$

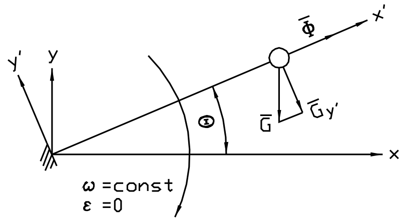

Fig. 2. Rotation of a flexible link in vertical plane with constant velocity

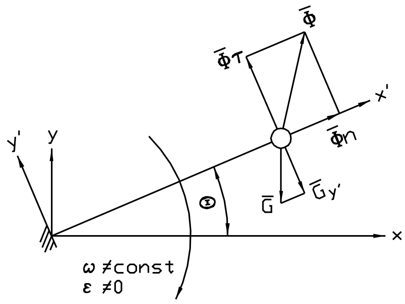

Fig. 3. Rotation of an elastic link in vertical plane with variable velocity

The equation of the elastic line with the initial conditions is as follows:

$$EI_{z'}\frac{d^2y'}{dx'^2} = -mg\cos\theta(l - x'), \quad x' = 0, \quad y' = 0; \quad \frac{dy'}{dx'} = 0. \tag{4}$$

Then, the first and the second integral are

$$EI_{z'}\frac{dy'}{dx'} = mg\cos\theta\,(\frac{(l-x')^2}{2} - \frac{l^2}{2}), \quad y' = -\frac{mg\cos\theta}{EI_{z'}}(l\frac{x'^2}{2} - \frac{x'^3}{6}). \tag{5}$$

In this way, an equation that describes the elastic line of the link at a fixed time moment is obtained. Then, during the whole motion, the elastic displacements can be described with the following system

$$\theta = \theta(t), \quad y' = -\frac{mg\cos\theta}{EI_{z'}}(l\frac{x'^2}{2} - \frac{x'^3}{6}). \tag{6}$$

Using the corresponding rotation matrix $R(rot.axis, angle)$, the elastic displacements in the absolute coordinate system seem like

$$^0\delta = R(z,\theta)'\delta, \tag{7}$$

where $^0\delta = [^0x\ ^0y]^T$ and $'\delta = ['x\ 'y]^T$ and the upper left index shows according to which coordinate system we have described the respective quantity.

CASE 2. Rotation of an Elastic Link in Vertical Plane with Variable Velocity.

This case (Fig. 3) is a similar one to CASE 1, but here the movement is accelerating and the angular velocity is not constant. The admissions are the same as in the previous case, but here we have introduced an acceleration constrains, namely ε is continuous and smooth, i.e. $\dot{\varepsilon}$ exists for every t and it is necessary the first derivative of the acceleration of the mass point to be fixed at every moment. The forces $\overline{\Phi}$ and $\overline{G}$ have vertical components in the movable coordinate system $Ox'y'z'$, hence they generate moments with respect to the hang point of the link. $\Phi_{y'}$ is nothing else but $\Phi_\tau = m\varepsilon l$, where $\varepsilon = \varepsilon(t)$ is a known function of the acceleration. The equation of the elastic line is

$$EI_{z'}\frac{d^2y'}{dx'^2} = (m\varepsilon l - mg\cos\theta)(l - x'). \tag{8}$$

We follow the same procedure given in CASE 1. After double integration and taking into account the initial conditions, which are the same as before, we obtain the system that describes the elastic displacements of the link throughout the whole motion:

$$\theta = \theta(t), \ \ \ddot{\theta} = \ddot{\theta}(t), \ \ y' = \frac{m\varepsilon l - mg\cos\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right). \tag{9}$$

CASE 3. Rotation of the Elastic Link in Horizontal Plane with Constant Velocity.

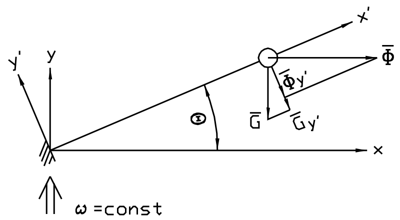

Fig. 4. Rotation of an elastic link in horizontal plane with constant velocity

In this case (Fig. 4) the motion is a constant rotation around the immovable axis Oy. Let us attach again a local coordinate system $Ox'y'z'$ to the link . The two acting forces are the gravity force $\overline{G}$ and the inertia force $\overline{\Phi}$. Moments are generated by their vertical components

$$G_{y'} = G\cos\theta = mg\cos\theta, \qquad \Phi_{y'} = \Phi\sin\theta = ma_O\sin\theta = m\omega^2 l\cos\theta\sin\theta,$$

where a_O is the centrifugal acceleration of the point mass. In this case θ does not depend on the time and appears to be constant during the whole motion.

The elastic beam, after reaching its maximum deformations, behaves itself as if it is a rigid body making pure rotation. The equation of the elastic line is

$$EI_z\frac{d^2y'}{dx'^2} = -\Phi_{y'}(l - x') - G_{y'}(l - x'). \tag{10}$$

This equation describes the form of the elastic line of the beam at a fixed time moment. But when $\omega = const$, then $\Phi = m\omega^2 h =$ const, i.e. the equation is valid for every time moment.

The solution of the equation of the elastic line at a fixed time moment is

$$y' = -\frac{mg\cos\theta + m\omega^2 l\cos\theta\sin\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right). \tag{11}$$

Due to the constant velocity of the link rotation, the above equation describes in the local coordinate system the displacements not only at a definite moment, but also during the whole motion.

CASE 4. Rotation of the Elastic Link in Horizontal Plane with Variable Velocity.

The case is a similar to the previous one, but now the angular velocity of the elastic link is not a constant quantity. The admissions are the same as in the previous case, but here we have introduced an acceleration constrains, namely ε is continuous and smooth, i.e. $\dot{\varepsilon}$ exists for every t, and it is necessary the first derivative of the acceleration of the mass point to be fixed at every moment. The link is related around the immovable axis Oy with a non-constant velocity, i.e. $\theta =$ const and $z \equiv z'$ only at the initial moment. During the motion axis Oy' describes a circular cone whose axis of symmetry coincides with axis Oy. The vertex angle of the cone is 2θ. If the movement is not accelerating, the case will coincide with the previous one. However, as a result of tangential acceleration, here we obtain an additional component of the inertia force. This component loads the link of bending in the plane that contains the axis Ox' and the acceleration $\overline{a}_\tau$ itself. Let us define the displacements of the point mass in this plane, and once we have found the displacements in the vertical plane, containing the gravity force $\overline{G}$ and centrifugal acceleration $\overline{a}_O$, we can obtain the total displacement. So that, the tangential component of the inertia force in the plain $Ox'y'$ is

$$\Phi_\tau = ma_\tau = m\varepsilon l. \tag{12}$$

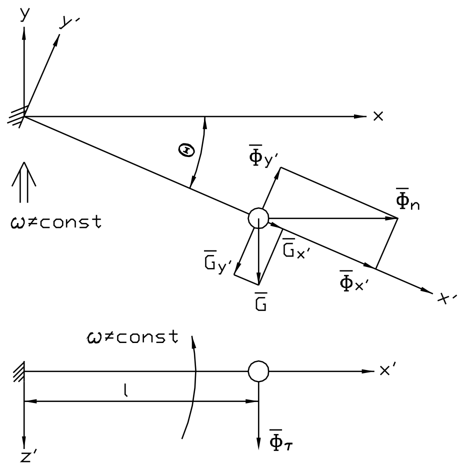

Fig. 5. Rotation of an elastic link in horizontal plane with variable velocity

The equation of the elastic line of the link in this plane at a fixed time moment with the initial conditions is

$$EI_{y'}\frac{d^2z'}{dx'^2} = m\varepsilon l(l - x'), \quad x' = 0, \quad z' = 0, \quad \frac{dz'}{dx'} = 0. \tag{13}$$

The solution is

$$z' = \frac{m\varepsilon l}{EI_{y'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right). \tag{14}$$

The form of the elastic line in the plain $Ox'y'$ may be obtained as in the previous case. The difference here, is that the ω is not constant. The form of the elastic line during the whole motion is described by the following system of equations

$$\begin{aligned} \theta &= \theta(t), \quad \dot{\theta} = \dot{\theta}(t), \quad \ddot{\theta} = \ddot{\theta}(t) \\ z' &= \frac{m\varepsilon l}{EI_{y'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right), \quad y' = -\frac{mg\cos\theta - m\omega^2 l\cos\theta\sin\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right). \end{aligned} \tag{15}$$

The total displacement and the direction cosines respectively are

$$'\delta = \sqrt{'\delta_{z'}^2 + '\delta_{y'}^2}, \quad \gamma = \frac{'\delta_{z'}}{\sqrt{'\delta_{z'}^2 + '\delta_{y'}^2}}, \quad \beta = \frac{'\delta_{y'}}{\sqrt{'\delta_{z'}^2 + '\delta_{y'}^2}}. \tag{16}$$

4. General Motion of a Manipulator. In this section we consider an open-loop kinematic chain (manipulator) that consists of n joints with one degree of freedom and $n+1$ links (the base is denoted with 0). The last link (link n) is flexible, the rest ones are rigid. A point mass is immovably connected to the free end of the last link. A coordinate system oriented according to the Denavit-Hartenberg notation is attached to each link. It is convenient the axes of the n-th coordinate system to be oriented along the principal inertia axes of the link. In case that the above mentioned system doesn't coincide with the respective one of link n oriented according to Denavit-Hartenberg formalism, the transformation matrix between the two coordinate systems has to be composed. Further on, let us suppose, that the both systems coincide. In this right-hand frame $(Ox'y'z')$ coinciding with coordinate system n, the elastic displacements are calculated. The movement of the link is calculated in the inertial coordinate system $Oxyz$, which is fixed to the base of a manipulator. Axis Ox' appears to be a tangent to the elastic line of the link in the hang point of the link after its deformation; the axis Oz' is perpendicular to the axis Ox' and oriented along the rotation (translation) joint axis. With respect to the local coordinate system of the elastic link, its joint end is immovable and does not complete elastic displacements, since it is connected to the previous link which is rigid. From the other hand, the actual displacements of the point mass are due to its absolute acceleration and its weight. This procedure is given below (elements of this procedure may be found also in [2], [10], [8]). It consists of the following steps:

1. Description of the geometrical and mechanical characteristics of the links like: size, mass of point mass, elasticity module of link n.

2. Forming of the motion laws of the links (joint variables) together with cycle of the manipulator (the duration of manipulator motion).

3. Composing of the transformation matrices in the form

$$(17)\qquad {}^{i-1}_{\ \ i}T = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & \alpha_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where θ, a, α, d are the Denavit-Hartenberg parameters.

4. Calculation of angular and linear velocities.

4.1. For joint $(i+1)$ rotational, they are

$$(18)\quad {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}Z_{i+1}, \quad {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\,({}^{i}v_i +^{i}\omega_i \times^{i} P_{i+1}).$$

4.2. For joint $(i+1)$ prismatic, they are

$$(19)\quad {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i, \quad {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\,({}^{i}v_i +^{i}\omega_i \times^{i} P_{i+1}) + \dot{d}_{i+1}{}^{i+1}Z_{i+1},$$

where we have denoted

${}^{i+1}_{\ \ i}R$ – the inverse rotation matrix with dimension (3×3);

${}^{i}P_{i+1}$ – the distance from the origin of coordinate system $\{i\}$ to the origin of coordinate system $\{i+1\}$;

${}^{i+1}Z_{i+1}$ – the unit vector along the axis Z, i.e. $\dot{\theta}_{i+1}\,{}^{i+1}Z_{i+1} = {}^{i+1}\begin{bmatrix} 0 & 0 & \dot{\theta}_{i+1} \end{bmatrix}^T$.

5. Calculation of angular and linear accelerations.

5.1. For joint $(i+1)$ rotational, they are

$$(20)\qquad {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\dot{\omega}_i +^{i+1}_{\ \ i} R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}{}^{i+1}Z_{i+1} + \ddot{\theta}_{i+1}{}^{i+1}Z_{i+1},$$

$$(21)\qquad {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{\ \ i}R\,[{}^{i}\dot{\omega}_i \times^{i} P_{i+1} +^{i}\omega_i \times ({}^{i}\omega_i \times^{i} P_{i+1}) +^{i}\dot{v}_i].$$

5.2. For joint $(i+1)$ prismatic, they are

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i, \tag{22} $$

$$ \begin{aligned} {}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R\,[{}^{i}\dot{\omega}_i \times^{i} P_{i+1} +^{i} \omega_i \times ({}^{i}\omega_i \times^{i} P_{i+1}) +^{i} \dot{v}_i] + \\ &\quad +2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}{}^{i+1}Z_{i+1} + \ddot{d}_{i+1}{}^{i+1}Z_{i+1}. \end{aligned} \tag{23} $$

6. The inertia force of the point mass is composed

$$ {}^{n}\Phi_n = m\,{}^{i+1}\dot{v}_{i+1}. \tag{24} $$

7. The gravity force of the point mass with respect to the coordinate system n is calculated

$$ {}^{n}G_n = \left(\prod_{i=0}^{n} {}^{i}_{i+1}R\right)^{T} {}^{0}G_n. \tag{25} $$

8. The differential equations of the elastic line in the both plains are given

$$ \begin{aligned} EI_{z'}\frac{d^2y'}{dx'^2} &= \sum M_{z'} = ({}'\Phi_{y'} + {}'G_{y'})(l - x'); \\ EI_{y'}\frac{d^2z'}{dx'^2} &= \sum M_{y'} = ({}'\Phi_{z'} + {}'G_{z'})(l - x'). \end{aligned} \tag{26} $$

9. The displacements in the both plains are found and the total displacement, as well as the direction cosines, are obtained

$$ {}'\delta = \sqrt{{}'\delta^2_{z'} + {}'\delta^2_{y'}}, \quad \gamma = \frac{{}'\delta_{z'}}{\sqrt{{}'\delta^2_{z'} + {}'\delta^2_{y'}}}, \quad \beta = \frac{{}'\delta_{y'}}{\sqrt{{}'\delta^2_{z'} + {}'\delta^2_{y'}}}. \tag{27} $$

10. The displacements with respect to the absolute coordinate system are

$$ {}^{0}P_n = \prod_{i=1}^{n} {}^{i-1}_{i}T\,{}^{n}P_n, \tag{28} $$

where ${}^{n}P_n$ is (4×1) vector and represents the elastic displacement of the characteristic point (for example the end-effector) from the last link n in time and ${}^{i-1}_{i}T$ is the (4×4) homogeneous transformation matrix that represents the description of frame $\{i\}$ relatively to the frame $\{i-1\}$.

5. Example. We consider a $2R$ plane manipulator (Fig. 6) consisting of two movable links and two rotational joints, and a point mass at the free end of the second link, having the following characteristics: length of link 1: $L[1] = 1\ m$; length of link 2: $L[2] = 0.5\ m$; mass of the point mass at the end of the link 2: $m = 5\ kg$; the section of link 2 is a circle with radius: $r = 0.01\ m$ and elasticity module of link 2 (steel): $E = 2.1\ 10^{11}\ Pa$.

The joint motion laws are expressed through five-degree polynomials in time interval $10s$.

$$ \theta[1](t) = -\frac{5t^2}{2} + \frac{(1840+20\pi)t^3}{2000} + \frac{(-2220-30\pi)t^4}{20000} + \frac{(880+12\pi)t^5}{200000}, \tag{29} $$

$$ \theta[2](t) = -\frac{\pi}{2} - \frac{5t^2}{2} + \frac{(1840+20\pi)t^3}{2000} + \frac{(-2220-30\pi)t^4}{20000} + \frac{(880+12\pi)t^5}{200000}. \tag{30} $$

The largest displacements are obtained at the end of the elastic link, i.e. at the point mass which coordinates are $(0.5, 0, 0)$ with respect to the frame $Ox'y'z'$.

Figure 7 shows the mode of the vertical components of the inertia force (continuous line) and of the gravity force of the point mass during the motion, i.e. the components

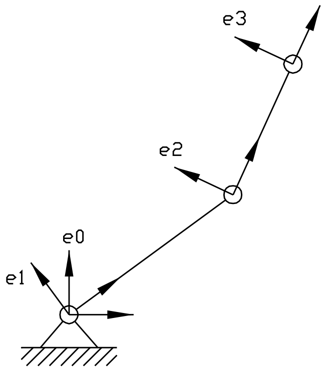

Fig. 6. $2R$ plane manipulator

$G_{y'}$ and $\Phi_{y'}$ depend on time t. Figure 8 shows the displacements of the elastic link (where the point mass is fixed) during the motion. The maximum positive and negative displacements of the point mass are respectively 0.0019 m when $t = 1.89$ s, and -0.002 m when $t = 5.8$ s with the link length 0.5 m. These results coincide with the assumption that the displacements are small.

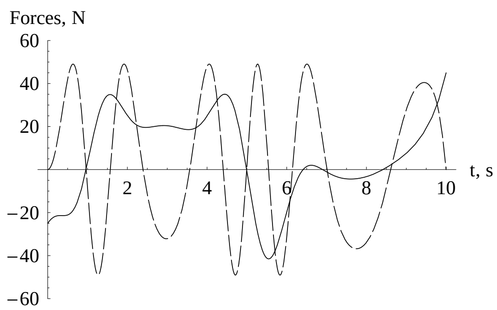

Fig. 7. $G_{y'}$ and $\Phi_{y'}$ depending on time t

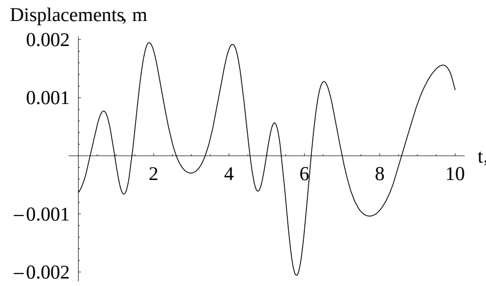

Fig. 8. Displacements of the elastic link (where the point mass is fixed) during the motion

6. Concluding Remarks. The considered algorithm can be used also in cases of more than one point masses arbitrary attached to the flexible link, as well as in cases of distributed loads, concentrated forces and moments. The mass of the link can be also included.

It is possible also, the contact points of the forces and concentrated moments, as well as the length of the link, to be changed with respect to the time. In its base part, the algorithm doesn't change. It is only necessary to correct the differential equation of the elastic line of link n.

The presented algorithm allows also a check of the maximum normal stresses of the elastic link to be included, for example the quotient of the maximum bending moment, and the moment of resistance to be calculated. However, it is necessary to take into account the force cyclicity, since only this kind of check could not be a strength criterion.

To minimize the vibrations of the elastic link, there shouldn't be jerk change of the acceleration, i.e. the acceleration to be fixed at each time moment, except in trivial cases, when the acceleration is constant or doesn't exist. The presented algorithm is valid only for cases of small displacements. When the displacements are large, it has to be taken into account that the contact point of the forces moves during the link deformation. Besides, Eq.(1) is obtained on the assumption that the quantity dy/dx is small compared to unity. In order to obtain large displacements, it is necessary the changes of the link curve to be considerable. But if the strains are not bigger than the limit of elasticity, that is possible only with small height of the section, then the link should be in the form of a thin band or a thin wire. The algorithms for generating laws of motion, kinematics of the manipulator and for calculating of the elastic displacements are modelled by software packages (Mathematica Package) in Mathematica 4.2. More information may be found in [10].

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ПРЕМЕСТВАНЕ НА ЕЛАСТИЧНО ЗВЕНО ОТ ОТВОРЕНА КИНЕМАТИЧНА ВЕРИГА (МАНИПУЛАТОР)

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Представен е подход за определяне на преместванията на еластично звено от отворена кинематична верига (манипулатор) при предположение, че преместванията са малки. Подходът е базиран на диференциалното уравнение на еластичната линия на огъната греда и на принципа на Даламбер. Законът за движение и уравнението на еластичната линия се записват и решават съвместно. Разгледани са случаите на най-простите движения на еластично звено с фиксирана тежка точка в свободния му край, както и общият случай, когато еластичното звено се явява последно в структурата на манипулационна система. Приведен е пример на двузвенен равнинен $2R$ манипулатор, второто звено на който е еластично.