МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2007
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2007
*Proceedings of the Thirty Sixth Spring Conference of
the Union of Bulgarian Mathematicians
St. Konstantin & Elena resort, Varna, April 2–6, 2007*

DIMENSION-RAISING FUNCTIONS AND MULTIPLE INTEGRALS

**Vladimir T. Todorov, Petar V. Stoev,
Mihail M. Konstantinov, Simeon T. Stefanov**

Let $G \subset \mathbb{R}^n$ be a connected domain in the $n-$dimensional space. It is well known that in fact G is a space-filling curve. At this juncture it is not astonishing that one can represent the multiple integral over G as a single one. Moreover, every metrizable compactum X is the image of the classical Cantor set C under a continuous map $\kappa : C \to X$. Then we can put $\int\limits_X f = \int\limits_C f \circ \kappa$.

1. Basic concepts and definitions. Teaching multiple integrals in the technical universities is not appropriate for many reasons. Perhaps, the most important one is the limited teaching time. Another reason is, probably, that different kinds of integrals are usually taught by (formally) different ways.

More precisely, the integrals are limits of Riemmann sums, and while for a double integral one requires one kind of sum, the triple integral is a limit of another type. Computing line and surface integrals also requires a special kind of Riemmann sums. This makes difficult teaching multiple integrals in technical universities. Here we offer an attempt to avoid the pointed obstacles by considering a space filling curves.

1.1. Space–filling curves or Peano curves are firstly described by Giuseppe Peano (1858—1932). Their ranges contain the entire 2-dimensional unit square (or the 3-dimensional unit cube). Intuitively, a “continuous curve” in the 2-dimensional plane or in the 3-dimensional space can be thought of as the “path of a continuously moving point”. To eliminate the inherent vagueness of this notion, Jordan in 1887 introduced the following rigorous definition which since has been adopted as the precise description of the notion of a “continuous curve”.

A curve (with endpoints) is a continuous function whose domain is the unit interval $[0, 1]$. In the most general form, the range of such a function may lie in an arbitrary topological space, but in most common cases, the range will lie in an Euclidean space such as the 2-dimensional plane (a “plane curve”) or the 3-dimensional space (a “space curve”). Sometimes, the curve is identified with the range or image of the function (the set of all possible values of the function), instead of the function itself. It is also possible to define curves without endpoints as continuous function on the real line (or on the open unit interval (0,1)).

Most of the well-known space-filling curves are constructed iteratively as the limit of a sequence of piecewise linear continuous curves. Approximation curves remain within a bounded portion of the n–dimensional space, but their lengths increase without bound. The space-filling curve is always self-intersecting, although the approximation curves in the sequence can be self-avoiding. Peano's original construction is self-contacting at all finite approximations.

1.2. Peano definition [1]**.** The original Peano curve maps the unit interval $[0,1]$ onto the unit square $[0,1]\times[0,1]$. The mapping is based on the ternary system, so that all digits t below may only take values 0, 1 or 2. First define the digit transformation k by $k(t)=2-t$ and denote by k^v, as usual, the v^{th} iterate of k. Peano's function $f(p)$ maps a ternary fraction $p=0.t_1t_2t_3...$ into a point $(x(p)(0.x_1x_2x_3...),y(p)(0.y_1y_2y_3...))$ in the unit square, where $x(p)$ and $y(p)$ are defined as

$$x(p)(0.t_1t_2t_3\dots)=0.t_1k^{t_2}(t_3)k^{t_2+t_4}(t_5)\dots$$

$$y(p)(0.t_1t_2t_3\dots)=0.k^{t_1}(t_2)k^{t_1+t_3}(t_4)\dots,$$

hence, $x_n=k^{t_2+\dots+t_{2n-2}}(t_{2n-1})$ (in particular $x_1=t_1$) and $y_n=k^{t_1+\dots+t_{2n-1}}(t_{2n})$ for $n=1,2,\dots$.

1.3. Perhaps a constriction like the above one should appear out of the ordinary to the students but it helps to avoid the descriptions of different kinds of Riemmann sums. In this section we describe a way to apply the above considerations for double integrals on the unit square K. The most convenient way to teach the plane-filling curve is, in our opinion the Hilbert curve. The endpoints of the Hilbert curve are $(0,0)$ and $(1,0)$ respectively. One can describe the Hilbert curve [2] as the range of the unit interval $[0,1]$ under the function $\vec{\kappa}$ which is a limit of a sequence of the piece-wise linear functions $\vec{\kappa}_n$. There exist multitude descriptions of $\vec{\kappa}_n$, including combinatorial ones. The most common way to teach them is to explain several steps of construction by picturing range of $\vec{\kappa}_n$ for the first several values of n.

The image of the interval $[0,1]$ under $\vec{\kappa}_1$, $\vec{\kappa}_2$ and $\vec{\kappa}_3$ is pictured on Figures 1, 2 and 3. figures.

The functions $\vec{\kappa}_n$ are well defined and it is not difficult to write an explicit description of $\vec{\kappa}_n$. Instead, we give two more steps in forming the Peano curve $\vec{\kappa}$.

1.4. For every integer n we divide the unit square in 4^n squares by dividing the sides of K in 2^n segments with length $\dfrac{1}{2^n}$. As it is shown at the figures, each $\vec{\kappa}_n:[0,1]\to K$ is a piece-wise linear vector function which walks through the points

$$\left\{\left(\frac{2k-1}{2^n},\frac{2l-1}{2^n}\right):k,l=1,2,\dots,2^{n-1}\right\}$$

in the above pointed way.

An important observation here is that for every $t\in[0,1]$ we have

$$\|\vec{\kappa}_{n+1}(t)-\vec{\kappa}_n(t)\|\le\frac{1}{2^n}$$

and hence the sequence $\{\vec{\kappa}_n\}$ is uniformly convergent to the continuous vector function $\vec{\kappa}$. It is not a problem to show that $\vec{\kappa}([0,1])=K$, since for an arbitrary $(x,y)\in K$ we have $\operatorname{dist}((x,y),\vec{\kappa}([0,1]))<\dfrac{1}{2^n}$.

1.5 Let, furthermore, $f:K\to\mathbb{R}$ be a continuous function and put $g(t)=f(\vec{\kappa}(t))$.

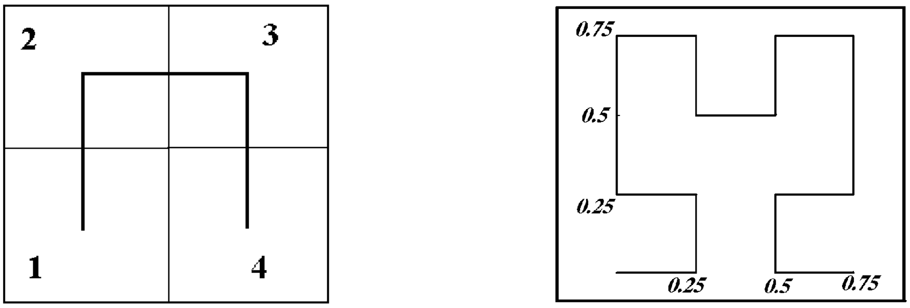

Fig. 1

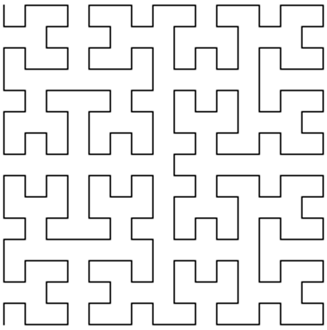

Fig. 2

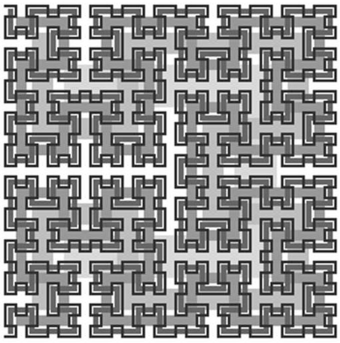

Fig. 3

Obviously g is a continuous function, hence it is integrable. Thus, we have

$$\int_0^1 g(t)\,\mathrm{d}t = \lim_{n\to\infty} \frac{1}{4^n} \sum_{p=1}^{4^n-1} g\left(\frac{p}{4^n}\right).$$

Note that from the choice of the Hilbert function it follows now that for each p one can find a pair (k,l) such that $\vec{\kappa}\left(\frac{p}{4^n}\right) = \left(\frac{k}{2^n}, \frac{l}{2^n}\right)$. Therefore,

$$g\left(\frac{p}{4^n}\right) = f\left(\vec{\kappa}\left(\frac{p}{4^n}\right)\right) = f\left(\frac{2k-1}{2^n}, \frac{2l-1}{2^n}\right)$$

for some unique pair $(2k-1, 2l-1)$. Thus, we obtain that

$$\int_0^1 g(t)\,\mathrm{d}t = \int_0^1 f(\vec{\kappa}(t))\,\mathrm{d}t = \lim_{n\to\infty} \frac{1}{4^n} \sum_{k=1}^{2^n-1} \sum_{l=1}^{2^n-1} f\left(\frac{k}{2^n}, \frac{l}{2^n}\right) = \iint_K f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

because f is a continuous function and, hence, integrable over K. Speaking more generally, the above considerations holds for every integrable in the sense of Riemmann function, defined on K.

1.6. In the same way one can define a triple integral by using space-filling curves. We give here the initial steps of the construction of the Peano 3-dimensional filling curve.

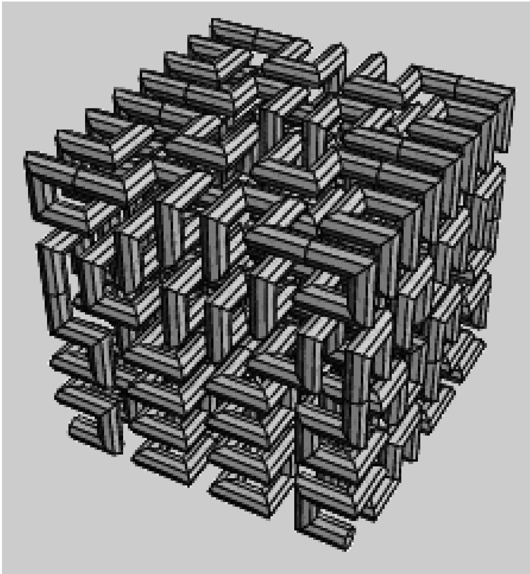

Fig. 4

2. A piece of research. We can use the Hahn-Mazurkiewicz theorem which gives the characterization of general continuous curves as formulated below:

A Hausdorff topological space is a continuous image of the unit interval if and only if it is a compact, connected, locally connected second-countable space.

Note. In many formulations of the Hahn-Mazurkiewicz theorem, "second-countable" is replaced by "metrizable". These two formulations are equivalent. In one direction a compact Hausdorff space is a normal space and, by the Uryson metrization theorem, second-countable then implies metrizable. Conversely, a compact metric space is second-countable.

It follows from the above, that we can compute any type of integrals on a compact metric space X in following way. First, we take some Peano curve $\kappa : [0,1] \to X$ and a measure μ on the interval $[0,1]$. In the second step we may consider the *image* of μ in X [3, p. 328] to obtain a measure ν in X. Clearly, one should have $\int\limits_X f = \int\limits_0^1 f \circ \kappa$.

2.1. If in the compact space X some measure μ is defined, then we may consider the pro-image of μ in $[0,1]$ and this also reduces the integral $\int\limits_X f$ to a definite integral over the interval $[0,1]$. One may continue by considering integrals on the Cantor set, but this matter is not a subject of discussion in the present note.

2.2. According to the theorem of Hurewicz [4], for every dimension-raising function $f : C \to X$ the set of the points $x \in X$, for which $\mathrm{card}(f^{-1}(y)) > 1$ is non-empty. One may prove that the pro-image of the measure μ in X is "good" if $\mathrm{card}(f^{-1}(y)) \leq \dim X + 1$ for every $x \in X$.

Finally one can prove the following assertion.

2.3. *If* $\vec{f} : I \to I^2$ *(*$i = [0,1]$*) is an arbitrary plane (or space)-filling curve then the* *pro-image of the Jordan (or Lebesgue) measure μ on I^2 is a measure of the same type if and only if $\mu_I(\vec{f}^{-1}(M)) = 0$ for every $M \subset I^2$ (or $M \subset I^3$) with $\mu_{I^j}(M) = 0$ for $j = 2, 3$.*

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V. Todorov, P. Stoev, M. Konstantinov, S. Stefanov
University of Architecture, Civil Engineering and Geodesy
1, Hr. Smirnenski Blvd
1046 Sofia, Bulgaria
e-mail: vtt_fte@uacg.bg, pvs_fte@uacg.bg, mmk_fte@uacg.bg

ПОВИШАВАЩИ РАЗМЕРНОСТТА ИЗОБРАЖЕНИЯ И КРАТНИ ИНТЕГРАЛИ

Владимир Т. Тодоров, Петър В. Стоев,
Михаил М. Константинов, Симеон Т. Стефанов

Нека $G \subset \mathbb{R}^n$ е свързана област в n-мерното Евклидово пространство. Добре известен факт е, че в този случай G е *крива*, т.е. непрекъснат образ на интервал. В тази бележка показваме, че от това следва, че кратните интеграли върху свързани множества могат да бъдат сведени към единични интеграли върху интервала $[0, 1]$.
По-общо, всеки метризуем компакт X е непрекъснат образ на класическото Канторово множество C посредством някакво непрекъснато изображение $\kappa : C \to X$. Ясно е, че тогава можем да дефинираме $\int\limits_X f = \int\limits_C f \circ \kappa$. В тази бележка обсъждаме възможността за преподаване на кратни интеграли в техническите университети по този малко екзотичен начин.