

## GEOMETRY OF NEUTRAL METRICS IN DIMENSION FOUR\*

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The purpose of this article is to review some recent results on the geometry of neutral signature metrics in dimension four and their twistor spaces. The following topics are considered: Neutral Kähler and hyperkähler surfaces, Walker metrics, Neutral anti-self-dual 4-manifolds and projective structures, Twistor spaces of neutral metrics.

**1. Introduction.** Riemannian and Lorentzian metrics in dimension four have been studied extensively for a long time in connection with various important problems in geometry and physics. The efforts were concentrated mainly on understanding of the corresponding Einstein equations which led to deep results in both fields. The metrics of neutral signature  $(+, +, -, -)$  appear also in many geometric and physics problems but have received less attention until recently. An impulse for the development of the geometry in this signature was the work of Ooguri and Vafa [39] who showed that this setting naturally arises in  $N = 2$  string theory. Moreover, it was realized that powerful techniques from the Seiberg-Witten theory and the theory of integrable systems can be successfully used to study Kähler-Einstein and self-dual metrics as well as the self-dual Yang-Mills equations in neutral signature (see [43], [24] and [15] and references therein).

At linear level, there are close analogues between the neutral signature and Riemannian metrics. For instance, in both cases the Hodge star operator is an involution of the bundle of two-forms, which leads to a splitting of the curvature operator into four irreducible components. Thus, the most significant classes of Riemannian metrics like Einstein and (anti) self-dual have natural counterparts in the neutral case. However, there are profound differences between the two geometries, both locally and globally. For example, in the neutral case the gauged-fixed self-duality equations are ultrahyperbolic, whereas in the Riemannian case they are elliptic which suggests that the neutral case is less rigid than the Riemannian one. Indeed, any Riemannian self-dual conformal structure must be real analytic, but this is not true for neutral structures.

In the present survey we review some recent results on the geometry of neutral signature metrics on 4-manifolds with emphasis on the differences with the Riemannian case.

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\*2000 Mathematics Subject Classification: : 53B30, 53C50.

**Key words:** Neutral metrics, neutral Kähler and hyperkähler surfaces, Walker metrics, twistor spaces.

\*\*Partially supported by CNRS-BAS joint research project *Invariant metrics and complex geometry*, 2008-2009.

In Sections 3 and 4 we present the results of Petean [43] and Kamada [29] about the existence of neutral Kähler and neutral hyperkähler metrics on compact complex surfaces.

Section 5 is devoted to the so-called Walker metrics which have interesting curvature properties and provide examples showing that some local integrability results related to the Goldberg conjecture [27] are not true for neutral metrics.

In Section 6 we describe the Dunajski-West [23] correspondence between the two dimensional projective structures and the anti-self-dual neutral 4-manifolds with a nontrivial null conformal Killing vector field.

Finally, in Section 7, we review some geometric and analytic results on the hyperbolic twistor spaces [11] and discuss their relation to the LeBrun-Mason twistor spaces of neutral self-dual manifolds [32].

This survey on the geometry of neutral metrics in dimension four is by no means comprehensive and it is in a sense complementary to [24] where the connection of neutral anti-self-dual conformal structures with integrable systems and twistor theory is underlined.

It is a pleasure to thank the Program Committee of the Thirty Seventh Spring Conference of the Union of Bulgarian Mathematicians for the kind invitation to deliver a lecture on geometry of neutral metrics.

**2. Preliminaries.** A pseudo-Riemannian metric on a smooth 4-manifold  $M$  is called *neutral* if it has signature  $(+, +, -, -)$ . In contrast to the Riemannian case the existence of such metrics on compact manifolds imposes topological restrictions since it is equivalent to the existence of a field of 2-planes. We refer to [34, 24] for more information.

Let  $M$  be an oriented 4-manifold with a neutral metric  $g$ . Then,  $g$  induces an inner product on the bundle  $\Lambda^2$  of bivectors by

$$\langle X_1 \wedge X_2, X_3 \wedge X_4 \rangle = \frac{1}{2} [g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)],$$

$X_1, \dots, X_4 \in TM$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_4$  be a local oriented orthonormal frame of  $TM$  with  $\|\mathbf{e}_1\|^2 = \|\mathbf{e}_2\|^2 = 1$ ,  $\|\mathbf{e}_3\|^2 = \|\mathbf{e}_4\|^2 = -1$ . As in the Riemannian case, the Hodge star operator  $*$  :  $\Lambda^2 \rightarrow \Lambda^2$  is an involution given by

$$*(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_4) = -\mathbf{e}_2 \wedge \mathbf{e}_3.$$

Denote by  $\Lambda_{\pm}$  the subbundles of  $\Lambda^2$  determined by the eigenvalues  $\pm 1$  of the Hodge star operator. Set

$$(1) \quad \begin{aligned} s_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4, & \bar{s}_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4, \\ s_2 &= \mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_4, & \bar{s}_2 &= \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4, \\ s_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3, & \bar{s}_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3. \end{aligned}$$

Then,  $\{s_1, s_2, s_3\}$  and  $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$  are local oriented orthonormal frames of  $\Lambda_-$  and  $\Lambda_+$  respectively with  $\|s_1\|^2 = \|\bar{s}_1\|^2 = 1$ ,  $\|s_2\|^2 = \|\bar{s}_2\|^2 = \|s_3\|^2 = \|\bar{s}_3\|^2 = -1$ .

Let  $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$  be the curvature operator of  $(M, g)$ . It is related to the curvature tensor  $R$  by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T); \quad X, Y, Z, T \in TM.$$

In this paper we adopt the following definition of the curvature tensor  $R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$ , where  $\nabla$  is the Levi-Civita connection of  $g$ . The curvature operator  $\mathcal{R}$  admits

an  $SO(2, 2)$ -irreducible decomposition

$$\mathcal{R} = \frac{\tau}{6}I + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-$$

similar to that in the 4-dimensional Riemannian case [8]. Here  $\tau$  is the scalar curvature,  $\mathcal{B}$  represents the traceless Ricci tensor,  $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$  corresponds to the Weyl conformal tensor, and  $\mathcal{W}_\pm = \mathcal{W}|_{\Lambda_\pm} = \frac{1}{2}(\mathcal{W} \pm *\mathcal{W})$ . The metric  $g$  is Einstein exactly when  $\mathcal{B} = 0$  and is conformally flat when  $\mathcal{W} = 0$ . It is said to be self-dual, resp. anti-self-dual, if  $\mathcal{W}_- = 0$ , resp.  $\mathcal{W}_+ = 0$ .

A neutral almost Hermitian structure on  $M$  consists of an almost complex structure  $J$  and a neutral metric  $g$  satisfying the compatibility condition  $g(JX, JY) = g(X, Y)$ . If the almost complex structure  $J$  is integrable (i.e. comes from a complex structure on  $M$ ), then the structure  $(g, J)$  is called Hermitian. As in the positive definite case any neutral almost Hermitian structure  $(g, J)$  determines a non-degenerate 2-form  $\Omega(X, Y) = g(JX, Y)$  but now  $\Omega$  is compatible with the opposite orientation of  $M$ . If  $\Omega$  is closed (i.e., it is a symplectic form), then the structure is said to be neutral almost Kähler and it is called neutral Kähler if, in addition, the almost complex structure  $J$  is integrable.

To describe the neutral almost Hermitian structures on  $(M, g)$  in terms of bivectors we identify  $\Lambda^2$  with the bundle of skew-symmetric endomorphisms of  $TM$  by assigning to each  $\sigma \in \Lambda^2$  the endomorphism  $J_\sigma$  on  $T_pM$ ,  $p = \pi(\sigma)$ , defined by

$$(2) \quad g(J_\sigma X, Y) = 2g(\sigma, X \wedge Y); X, Y \in T_pM.$$

Then, the local sections  $s_1, s_2, s_3$  of  $\Lambda_-$ , defined by (1), determine local endomorphisms  $J_1, J_2, J_3$  of  $TM$  that satisfy the relations

$$(3) \quad J_1^2 = -J_2^2 = -J_3^2 = -Id, \quad J_2J_3 = -J_3J_2 = Id$$

of the imaginary units of the paraquaternionic algebra (split quaternions). Hence, the set of all

$$J = y_1J_1 + y_2J_2 + y_3J_3, \quad y_1^2 - y_2^2 - y_3^2 = 1$$

describes the local almost complex structures compatible with  $g$  and the orientation of  $M$ .

If there are three global integrable structures  $J_1, J_2, J_3$  on  $M$  satisfying the relations (3), then the structure  $(g, J_1, J_2, J_3)$  is called neutral hyperhermitian. When additionally the 2-forms  $\Omega_i(X, Y) = g(J_iX, Y)$  are closed, the hyperhermitian structure is called neutral hyperkähler. It is well-known [30] that the neutral hyperhermitian metrics are self-dual, whereas the neutral hyperkähler metrics are self-dual and Ricci-flat.

**3. Neutral Kähler surfaces.** It is well-known that a compact complex surface admits a Kähler metric if and only if its first Betti number is even (see e.g. [7]). The analogous problem for neutral Kähler metrics is not yet solved completely. Using the Seiberg-Witten theory, Petean [43] proved the following:

**Theorem 1.** *Let  $(M, g, J)$  be a compact neutral Kähler surface and  $k(M, J)$  its Kodaira dimension.*

(i) *If  $k(M, J) = -\infty$ , then  $(M, J)$  is either a ruled surface, or a surface of class  $VII_0$  with no global spherical shell and with positive even second Betti number.*

(ii) *If  $k(M, J) = 0$ , then  $(M, J)$  is either a hyperelliptic surface, a primary Kodaira surface or a complex torus.*

(iii) If  $k(M, J) = 1$ , then  $(M, J)$  is a minimal properly elliptic surface with zero signature.

(iv) If  $k(M, J) = 2$ , then  $(M, J)$  is a minimal surface of general type with nonnegative even signature.

As it is noted in [30], all surfaces listed in (ii) and (iii) as well as the ruled surfaces in (i) do admit neutral Kähler metrics but the existence of such metrics in the remaining cases is still open. Moreover, there is a conjecture by Nakamura [37] which claims that there are no surfaces satisfying the conditions in the second case of (i).

Now we turn to the existence of self-dual neutral Kähler metrics noting that in this case the self-duality is equivalent to the vanishing of the scalar curvature. The following result of Kamada [30] was obtained by means of Theorem 1.

**Theorem 2.** *Let  $(M, g, J)$  be a compact self-dual neutral Kähler surface and let  $\tau(M)$  and  $c_1(M, J)$  be its signature and first Chern class, respectively. Then,  $(M, J)$  is one of the following:*

(i) *A surface of class  $VII_0$  with no global spherical shell and with positive even second Betti number, if  $c_1(M, J)^2 < 0$ .*

(ii) *A minimal surface of general type with even signature, if  $c_1(M, J)^2 > 0$  and  $\tau(M) > 0$ .*

(iii) *A Hirzebruch surface or a minimal surface of general type uniformized by the bidisc, if  $c_1(M, J)^2 > 0$  and  $\tau(M) = 0$ .*

(iv) *A hyperelliptic surface, a primary Kodaira surface, a complex torus or a minimal properly elliptic surface with zero signature, if  $c_1(M, J)^2 = 0$ .*

As it is noted by Kamada [30], it is not known whether the surfaces in (ii) and the properly elliptic surfaces with zero signature admit neutral self-dual Kähler metrics.

The following theorem of Petean [43] gives a complete classification of compact complex surfaces admitting neutral Ricci-flat Kähler metrics as well as an almost complete classification of surfaces that admit neutral Kähler-Einstein metrics with non-zero scalar curvature.

**Theorem 3.** *Let  $(M, g, J)$  be a compact neutral Kähler-Einstein surface. Then,  $(M, J)$  is one of the following:*

(i) *A complex torus.*

(ii) *A hyperelliptic surface.*

(iii) *A primary Kodaira surface.*

(iv) *A minimal ruled surface over a curve of genus  $g \geq 2$ .*

(v) *A minimal surface of class  $VII_0$  with no global spherical shell, and with even and positive second Betti number.*

*If  $(M, J)$  admits a neutral Ricci-flat Kähler metric, then it is as in (i), (ii) or (iii).*

Petean [43] constructed also examples of neutral Kähler – Einstein metrics on the surfaces of type (i), (ii), (iii) and on most surfaces of type (iv). We note that Petean's metrics on a complex torus are actually hyperkähler and depend on an arbitrary smooth function of two real variables. This shows that the moduli spaces of neutral Ricci-flat Kähler metrics can be highly nontrivial and different from those in the positive definite case.

**4. Neutral hyperkähler surfaces.** As is well-known (c.f. [8]), any compact hyperkähler surface is either a complex torus with a flat metric, or a  $K3$ -surface with Calabi-Yau metric. In contrast, H. Kamada [29] proved that neutral hyperkähler structures can exist either on a complex torus, or on a primary Kodaira surface and obtained a description of all such structures on both types of surfaces. Here we provide his result for primary Kodaira surfaces.

Consider the affine transformations  $\rho_i(z_1, z_2) = (z_1 + a_i, z_2 + \bar{a}_i z_1 + b_i)$  of  $\mathbb{C}^2$ , where  $a_i, b_i$ ,  $i = 1, 2, 3, 4$  are complex numbers such that  $a_1 = a_2 = 0$ ,  $Im(a_3 \bar{a}_4) = b_1$ . Then,  $\rho_i$  generate a group  $G$  of affine transformations acting freely and properly discontinuously on  $\mathbb{C}^2$ . The quotient space  $\mathbb{C}^2/G$  is called a primary Kodaira surface.

Note also [28, 29] that any neutral hyperkähler structure is determined by three symplectic forms  $(\Omega_1, \Omega_2, \Omega_3)$  satisfying the relations

$$-\Omega_1^2 = \Omega_2^2 = \Omega_3^2, \quad \Omega_l \wedge \Omega_m = 0, \quad l \neq m.$$

In these terms the result of Kamada [29] is the following:

**Theorem 4.** *For any neutral hyperkähler structure on a primary Kodaira surface  $(M, J)$  there are complex coordinates  $(z_1, z_2)$  of  $\mathbb{C}^2$  such that the structure is given by the following symplectic forms:*

$$\Omega_1 = Im(dz_1 \wedge d\bar{z}_2) + iRe(z_1)dz_1 \wedge d\bar{z}_1 + (i/2)\partial\bar{\partial}\phi,$$

$$\Omega_2 = Re(e^{i\theta} dz_1 \wedge d\bar{z}_2), \quad \Omega_3 = Im(e^{i\theta} dz_1 \wedge d\bar{z}_2),$$

where  $\theta$  is a real constant and  $\phi$  is a smooth function on  $M$  such that:

$$(4) \quad 4i(Im(dz_1 \wedge d\bar{z}_2) + iRe(z_1)(dz_1 \wedge d\bar{z}_1)) \wedge \partial\bar{\partial}\phi = \partial\bar{\partial}\phi \wedge \partial\bar{\partial}\phi$$

As it is proved by Kamada [29], the neutral hyperkähler metric on  $M$  defined by  $(\Omega_1, \Omega_2, \Omega_3)$  is flat if and only if the function  $\phi$  is constant. On the other side, any primary Kodaira surface is a toric bundle over an elliptic curve and the pull-back of any smooth function on the base curve gives a solution to (4). It follows that the moduli space of neutral hyperkähler structures on a primary Kodaira surface is infinite dimensional which is in sharp contrast with the positive definite case. We refer to [30] for analogous description of neutral hyperkähler structures on complex tori which are a product of elliptic curves.

**5. Walker metrics.** A basic problem in almost Hermitian geometry is to relate properties of an almost Hermitian structure  $(g, J)$  to the curvature of the metric  $g$ . For example, the well-known Goldberg conjecture [27] claims that a compact almost Kähler manifold is Kähler provided the metric  $g$  is Einstein. This conjecture was confirmed by Sekigawa [44] in the case of non-negative scalar curvature but it is still far from being solved in the negative case. We refer to the survey [4] for an update on the integrability of almost Kähler structures. Another integrability result related to the curvature properties of a manifold is the Riemannian version of the well-known Goldberg-Sacks theorem in General Relativity. It says that an oriented Einstein 4-manifold admits locally a compatible complex structure if and only if the spectrum of the positive Weyl tensor is degenerate [38]. We refer to [2] for generalizations of this result in the Riemannian setting and to [3] for analogous results for arbitrary pseudo-Riemannian 4-manifolds.

In this section we consider a special class of neutral metrics having interesting cur-

vature properties and provide examples showing that some integrability results in the Riemannian case are not true for neutral metrics.

A *Walker manifold* is a triple  $(M, g, \mathcal{D})$  where  $M$  is a smooth manifold,  $g$  an indefinite metric and  $\mathcal{D}$  a parallel null distribution. Of special interest are those manifolds admitting a field of null planes of maximum dimension. Since the dimension of a null plane is not greater than  $\dim(M)/2$ , the lowest possible case is that of neutral metrics in dimension four admitting a field of parallel null 2-planes.

Observe that Walker metrics appear as the underlying structure of several specific pseudo-Riemannian structures such as:

- Hypersymplectic structures [28];
- Einstein hypersurfaces in indefinite space forms [33];
- Indefinite Kähler Lie algebras [42];
- 2-step nilpotent Lie groups with degenerate center [14].

Note also that Walker metrics play a distinguished role in investigating the holonomy of indefinite metrics (see for example [9] and the references therein).

Recall that, by a result of Walker [45], for every Walker metric  $g$  on a 4-manifold  $M$ , there exist local coordinates  $(x, y, z, t)$  around any point of  $M$  such that the matrix of  $g$  in these coordinates has the following form

$$(5) \quad g_{(x,y,z,t)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}$$

for some smooth functions  $a$ ,  $b$  and  $c$ . In what follows, we always assume that the Walker metrics under consideration are given by (5).

**5.1. Self-dual Walker metrics.** The special coordinates mentioned above have been used in [22, 21] to obtain local description of self-dual and Einstein self-dual Walker metrics.

**Theorem 5.** *A Walker metric is self-dual if and only if the functions  $a, b, c$  have the form*

$$\begin{aligned} a &= x^2yA + x^3B + x^2C + 2xyD + xE + yF + G, \\ b &= xy^2B + y^3A + y^2K + 2xyL + xM + yN + P, \\ c &= x^2yB + xy^2A + x^2L + y^2D + \frac{1}{2}xy(C + K) + xQ + yR + S, \end{aligned}$$

where  $A, B, C$ , etc. are arbitrary smooth functions of  $(z, t)$ .

**Theorem 6.** *A Walker metric is Einstein and self-dual if and only if the functions  $a, b, c$  have the form*

$$\begin{aligned} a &= x^2K + xE + yF + G, \\ b &= y^2K + xM + yN + P, \\ c &= xyK + xQ + yR + S, \end{aligned}$$

where  $K$  is a constant and  $E, F, G$ , etc. are arbitrary smooth functions of  $(z, t)$  satisfying

the following PDE's:

$$\begin{aligned} 2R_z - 2F_t &= FQ + R^2 + KG - RE - FN, \\ E_t + N_z - R_t - Q_z &= FM - QR + KS, \\ 2Q_t - 2M_z &= MR + Q^2 + KP - EM - QN. \end{aligned}$$

The next result provides large families of neutral metrics whose hyperbolic twistor spaces are isotropic Kähler but non-Kähler (see Theorem 18, (iii))

**Theorem 7.** *A Walker metric is self-dual and satisfies the conditions  $\mathcal{B}^2|_{\Lambda_-} = 0$  if and only if the functions  $a, b, c$  have the form*

$$\begin{aligned} a &= x^2K + xE + yF + G, \\ b &= y^2K + xM + yN + P, \\ c &= xyK + xQ + yR + S, \end{aligned}$$

where  $K, E, F$ , etc. are arbitrary smooth functions of  $(z, t)$ . In this case the metric has constant scalar curvature if and only if  $K$  is a constant.

**5.2. Hyperhermitian Walker metrics.** Let  $g$  be a Walker metric on  $\mathbb{R}^4$  having the form (5). Then, an orthonormal frame of  $T\mathbb{R}^4$  can be specialized by using the canonical coordinates as follows:

$$\begin{aligned} \mathbf{e}_1 &= \frac{1-a}{2}\partial_x + \partial_z, & \mathbf{e}_2 &= \frac{1-b}{2}\partial_y + \partial_t - c\partial_x, \\ \mathbf{e}_3 &= -\frac{1+a}{2}\partial_x + \partial_z, & \mathbf{e}_4 &= -\frac{1+b}{2}\partial_y + \partial_t - c\partial_x. \end{aligned}$$

Then, the global sections  $s_1, s_2, s_3$  of  $\Lambda^2$ , defined by (1), determine via (2) a neutral almost hyperhermitian structure  $(g, J_1, J_2, J_3)$  on  $\mathbb{R}^4$  which is called proper [17].

Recall that an indefinite almost Hermitian structure  $(g, J)$  is said to be *isotropic Kähler*, if  $\|\nabla J\|^2 = 0$ . Isotropic Kähler structures were first investigated in [26] in dimension four and subsequently in [10] in dimension six. It is shown in [16] that the neutral almost Hermitian structure  $(g, J_1)$  on a Walker 4-manifold is isotropic Kähler. Moreover, we have the following [17]:

**Theorem 8.** *Any proper almost hyperhermitian structure  $(g, J_1, J_2, J_3)$  on a Walker 4-manifold satisfies  $\|\nabla J_i\|^2 = \|d\Omega_i\|^2 \|\delta\Omega_i\|^2 = \|N_i\|^2 = 0$ , where  $N_i$  is the Nijenhuis tensor of  $J_i$ ,  $i = 1, 2, 3$ .*

Examples of compact isotropic Kähler structures can be constructed on 4-tori taking  $a, b$  and  $c$  in (5) to be periodic functions on  $\mathbb{R}^4$ . Note also that in general the isotropic Kähler structure  $(g, J_1)$  is neither Kähler nor symplectic.

The next two results are proved in [17].

**Theorem 9.** *The structure  $(g, J_1, J_2, J_3)$  is neutral hyperhermitian if and only if the functions  $a, b$  and  $c$  have the form*

$$\begin{aligned} a &= x^2K + xP + \xi, \\ b &= y^2K + yT + \eta, \\ c &= xyK + \frac{1}{2}xT + \frac{1}{2}yP + \gamma, \end{aligned}$$

where the capital and Greek letters stand for arbitrary smooth functions of  $(z, t)$ .

**Theorem 10.** *The structure  $(g, J_1, J_2, J_3)$  is neutral hyperkähler if and only if the functions  $a, b$  and  $c$  do not depend on  $x$  and  $y$ .*

In particular, the above theorem shows that the metrics considered by Petean [43] are all neutral hyperkähler and, hence, self-dual and Ricci-flat. Moreover, this theorem together with the Petean's classification of neutral Ricci-flat Kähler surfaces (see Theorem 3) leads to the following description of compact neutral 4-manifolds with two parallel, orthogonal and null vector fields [20]:

**Theorem 11.** *Let  $(M, g)$  be a compact oriented neutral 4-manifold with two parallel, orthogonal and null vector fields. Then,  $M$  is diffeomorphic to a torus or to a primary Kodaira surface and the metric  $g$  is neutral hyperkähler.*

**5.3. Hermitian Walker metrics.** The almost complex structure  $J_1$  defined above was introduced by Matsushita [35] and called the *proper* almost complex structure of the Walker metric  $g$ . Further, we denote it by  $J$  and note that the structure  $(g, J)$  is:

- Hermitian if and only if  $a_x - b_x = 2c_y$ ,  $a_y - b_y = -2c_x$ ;
- Kähler if and only if  $a_x = -b_x = c_y$ ,  $a_y = -b_y = -c_x$ .

Moreover, the following results is proved in [17].

**Theorem 12.** *The structure  $(g, J)$  is Hermitian and self-dual if and only if*

$$a = x^2K + xyL + xP + yQ + \xi,$$

$$b = y^2K - xyL + xS + yT + \eta,$$

$$c = \frac{1}{2}(y^2 - x^2)L + xyK - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma,$$

where all capital and Greek letters are arbitrary smooth functions of  $(z, t)$ .

**Theorem 13.** *The structure  $(g, J)$  is Kähler self-dual if and only if*

$$a = xyL + xP + yQ + \xi,$$

$$b = -xyL - xP - yQ + \eta,$$

$$c = \frac{1}{2}(y^2 - x^2)L - xQ + yP + \gamma.$$

**Theorem 14.** *The structure  $(g, J)$  is Kähler-Einstein if and only if*

$$a = \kappa(x^2 - y^2) + xP + yQ + \xi,$$

$$b = \kappa(y^2 - x^2) - xP - yQ - \xi + \frac{1}{\kappa}(P_z - Q_t),$$

$$c = 2\kappa xy - xQ + yP + \gamma,$$

or

$$a = xP + yQ + \xi,$$

$$b = -xP - yQ + \eta,$$

$$c = -xQ + yP + \gamma,$$

where in the last case  $P_z = Q_t$ . Here  $\kappa$  is a non-zero constant and all capital and Greek letters are smooth functions of  $(z, t)$ . In the first case  $\tau = 8\kappa$ , in the second one  $\tau = 0$ .



The above theorem are used in [17] to construct examples of neutral almost Hermitian Einstein metrics showing that an integrability result of Kirchberg [31] in the positive definite case is not valid in the neutral setting

**5.4. Almost Kähler Walker metrics.** Although the Goldberg conjecture [27] mentioned above is of global nature, it is already known that some additional curvature conditions suffice to show the integrability of the almost complex structure at the local level. For instance, in dimension four, Einstein almost Kähler metrics which are  $*$ -Einstein are necessarily Kähler [40]. (The  $*$ -Einstein condition can be replaced by the second Gray curvature identity or the anti-self-duality condition and the integrability still follows [4]). It is also well-known [5, 40, 41] that any almost Kähler metric of constant sectional curvature is Kähler (and flat).

Our purpose here is to exhibit large families of neutral strictly almost Kähler Einstein metrics showing that the results mentioned above are not true for neutral metrics. Their construction is based on the following result [16]:

**Theorem 15.** *A proper structure  $(g, J)$  is strictly almost Kähler Einstein if and only if the functions  $a, b$  and  $c$  have the form*

$$a = xP + yQ + \xi,$$

$$b = -xP - yQ + \eta,$$

$$c = xU + yV + \gamma,$$

where all capital and Greek letters are smooth functions of  $(z, t)$  satisfying the following PDE's:

$$2(V_z - Q_t) = V^2 - VP + Q^2 + UQ,$$

$$2(P_z + U_t) = P^2 - VP + U^2 + UQ,$$

$$Q_z + U_z - P_t + V_t = PQ + UV,$$

and  $(V - P)^2 + (U + Q)^2 \neq 0$ .

An interesting consequence of this result (compare with the Riemannian case) is the following:

**Corollary 16.** *Any proper strictly almost Kähler – Einstein structure on a Walker 4-manifold is self-dual, Ricci flat and  $*$ -Ricci flat.*

**Example.** For any constants  $p, q, r$  with  $p^2 + q^2 \neq 0$ , set

$$a = -\frac{2(px - qy)}{pz + qt + r}, \quad b = -\frac{2(px + qy)}{pz + qt + r}.$$

It is shown in [16] that the proper almost Hermitian structure  $(g, J)$ , defined by the functions  $a, b$  and  $c = 0$ , is strictly almost Kähler and the metric  $g$  is flat.

Finally, let us note that we do not know examples of compact of neutral strictly almost Kähler–Einstein 4 manifolds. However, an indenfinite Ricci-flat almost Kähler metric on 8-dimentional torus is recently reported in [36].

**6. Neutral anti-self-dual 4-manifolds and projective structures.** In [23] Dunajski and West gave a local classification of neutral anti-self-dual 4-manifolds admitting a nontrivial null conformal Killing vector field. It is based on the observation that the self-dual and anti-self-dual null plane distributions, associated to such a vector field, are

integrable. Then, their main result is that the two dimensional leaf space of the foliation determined by the anti-self-dual plane distribution is equipped with a canonical projective structure, i.e. a class of torsion-free connections with the same unparametrized geodesics. Conversely, any projective structure on a 2-dimensional surface gives rise to a neutral anti-self-dual 4-manifold with null conformal Killing vector field. The explicit form of this correspondence is given by the following:

**Theorem 17.** *Let  $(M, [g], K)$  be a neutral anti-self-dual conformal structure with null conformal Killing vector field  $K$ . Then, around any point where  $K \neq 0$  there exist local coordinates  $(\phi, x, y, z)$  and  $g \in [g]$  such that  $K = \frac{\partial}{\partial \phi}$  and  $g$  has one of the following two forms, according to whether the twist  $\mathbb{K} \wedge d\mathbb{K}$  vanishes, or not (here  $\mathbb{K}$  is the one-form defined by  $\mathbb{K} := g(K, \cdot)$ ):*

(i) *If  $\mathbb{K} \wedge d\mathbb{K} = 0$ , then*

$$g = (d\phi + (zA_3 - Q)dy)(dy - \beta dx) - (dz - (z(-\beta_y + A_1 + \beta A_2 + \beta^2 A_3))dx - (z(A_2 + 2\beta A_3) + P)dy)dx,$$

where  $A_1, A_2, A_3, \beta, P, Q$  are arbitrary functions of  $(x, y)$ .

(ii) *If  $\mathbb{K} \wedge d\mathbb{K} \neq 0$ , then*

$$g = \left( d\phi + A_3 \frac{\partial G}{\partial z} dy + \left( A_2 \frac{\partial G}{\partial z} + 2A_3 \left( z \frac{\partial G}{\partial z} - G \right) - \frac{\partial^2 G}{\partial z \partial y} \right) dx \right) (dy - z dx) - \frac{\partial^2 G}{\partial z^2} dx (dz - (A_0 + zA_1 + z^2 A_2 + z^3 A_3) dx),$$

where  $A_0, A_1, A_2, A_3$  are arbitrary functions of  $(x, y)$ , and  $G$  is a function of  $(x, y, z)$  satisfying the following PDE:

$$\left( \frac{\partial}{\partial z} + \frac{\partial}{\partial y} + (A_0 + zA_1 + z^2 A_2 + z^3 A_3) \frac{\partial}{\partial z} \right) \frac{\partial^2 G}{\partial z^2} = 0.$$

The above two forms of the metric  $g$  are related to the two dimensional projective structures in the following way. The two geodesics equations for any connection in such a projective class can be reduced, by eliminating the affine parameter, to one second order ODE of the following form:

$$\frac{d^2 y}{dx^2} = A_3(x, y) \left( \frac{dy}{dx} \right)^3 + A_2(x, y) \left( \frac{dy}{dx} \right)^2 + A_1(x, y) \left( \frac{dy}{dx} \right) + A_0(x, y).$$

The functions  $A_i$  can be expressed in terms of the connection coefficients and do not depend on the particular choice of the connection. These are  $A_i$ 's which appear in the theorem above, where in the first case we have  $A_0 = \beta_x + \beta \beta_y - \beta A_1 - \beta^2 A_2 - \beta^3 A_3$ .

The relation between the anti-self-dual neutral 4-manifolds with a nontrivial null conformal Killing vector field and the two dimensional projective structures found by Dunajski and West, was generalized by Calderbank [13] for neutral anti-self-dual 4-manifolds admitting an integrable anti-self-dual null distribution instead of a null conformal Killing vector field. We should also mention the recent work by Bryant, Dunajski and Eastwood [12] on two dimensional metrisable projective structures which suggests the existence of a conformal invariant on neutral anti-self-dual manifolds, which vanishes when the conformal class contains a neutral Kähler metric with conformal null symmetry.

**7. Twistor spaces.** One can construct a twistor space of an oriented 4-manifold with a neutral metric as in the Riemannian case [11] (see also [1] where the pseudo-sphere  $SO(3, 2)/SO(2, 2)$  with the standard invariant neutral metric is discussed). These twistor spaces are called hyperbolic in [11], since their fibre is a two sheeted hyperboloid.

In this section we discuss the hyperbolic twistor spaces as well as their relation to the LeBrun-Mason twistor spaces of neutral self-dual manifolds whose fibre is  $\mathbb{CP}^1$  [32].

**7.1. Hyperbolic twistor spaces.** Let  $M$  be an oriented 4-manifold with a neutral metric  $g$ . Then, the hyperbolic twistor space  $\mathcal{Z}$  of  $M$  is defined to be the bundle on  $M$  consisting of all unit bivectors in  $\Lambda_-$ . It can be identified via (2) with the space of all complex structures on the tangent spaces of  $M$  compatible with its metric and orientation. Note that the fiber of the twistor bundle  $\pi : \mathcal{Z} \rightarrow M$  is the two sheeted hyperboloid

$$H = \{(y_1, y_2, y_3) \in \mathbb{R}^3 / -y_1^2 + y_2^2 + y_3^2 = -1\}$$

and we can think of the  $y_1 > 0$  branch as one of the standard models of the hyperbolic plane. Further, we consider the hyperboloid  $H$  with the complex structure  $S$  determined by the restriction to  $H$  of the metric  $-dy_1^2 + dy_2^2 + dy_3^2$ , i.e.  $SV = y \times V$  for  $V \in T_y H$ , where  $\times$  is the vector cross product on  $\mathbb{R}^3$  defined by means of the paraquaternionic algebra.

The Levi-Civita connection of  $g$  gives rise to a splitting  $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$  of the tangent bundle of  $\mathcal{Z}$  into horizontal and vertical components and we consider the vertical space  $\mathcal{V}_\sigma$  at  $\sigma \in \mathcal{Z}$  as the orthogonal complement of  $\sigma$  in  $\Lambda_-$ . As in the Riemannian case [6, 25], we define two almost complex structures  $\mathcal{J}_1$  and  $\mathcal{J}_2$  on  $\mathcal{Z}$  by

$$\mathcal{J}_n V = (-1)^{n-1} \sigma \times V, \quad V \in \mathcal{V}_\sigma,$$

$$\pi_*(\mathcal{J}_n A) = J_\sigma(\pi_* A), \quad A \in \mathcal{H}_\sigma,$$

where  $J_\sigma$  is the complex structure on  $T_p M$ ,  $p \in \pi(\sigma)$ , defined by (2).

The metric  $g$  on the bundle  $\pi : \Lambda^2 \rightarrow M$  induced by the metric of  $M$  is negatively definite on the fibres of  $\mathcal{Z}$  and we adopt the metric  $\langle \cdot, \cdot \rangle = -g$  on  $\Lambda^2$ . Setting  $h_t = \pi^* g + t \langle \cdot, \cdot \rangle$  for any real  $t \neq 0$  we get a 1-parameter family of pseudo-Riemannian metrics on  $\mathcal{Z}$  compatible with the almost complex structures  $\mathcal{J}_1$  and  $\mathcal{J}_2$ . The almost Hermitian structures  $(h_t, \mathcal{J}_k)$ ,  $t \neq 0, k = 1, 2$  is studied in [10, 11] where the following results is proved:

**Theorem 18.** *On the hyperbolic twistor space  $\mathcal{Z}$  of an oriented 4-manifold  $M$  with a neutral metric  $g$  we have the following:*

- (i) *The almost complex structure  $\mathcal{J}_1$  is integrable if and only if the metric  $g$  is self-dual. The almost Hermitian structure  $(\mathcal{J}_1, h_t)$  is indefinite Kähler if and only if  $g$  is an Einstein and self-dual metric with constant scalar curvature  $\tau = -12/t$ .*
- (ii) *The almost complex structure  $\mathcal{J}_2$  is never integrable. The almost Hermitian structure  $(\mathcal{J}_2, h_t)$  is indefinite almost Kähler (resp. nearly Kähler) if and only if the metric  $g$  is Einstein, self-dual and  $\tau = 12/t$  (resp.  $\tau = -6/t$ ).*
- (iii) *The almost Hermitian structure  $(h_t, \mathcal{J}_k)$  is isotropic Kähler if and only if  $k = 1$ ,  $g$  is a self-dual metric with constant scalar curvature  $\tau = -12/t$  and  $\mathcal{B}^2|_{\Lambda_-} = 0$ .*

Note that the values of the scalar curvature appearing in Theorem 18 for  $t > 0$  are the negatives of what one has for the usual twistor space [19]. This sign change is due to our choice of metric on  $\Lambda^2$ .

Statement (iii) of the above theorem motivates the study of self-dual neutral metrics of constant scalar curvature and two-step nilpotent Ricci operator  $\mathcal{B}$ . Note that it follows from Theorem 7 that this class of neutral metrics is strictly larger than the class of neutral self-dual Einstein metrics. The next example shows that it contains also neutral conformally flat metrics of nonconstant sectional curvature.

**Example.** Let  $G$  be a Lie group whose Lie algebra has a basis  $\{E_1, E_2, E_3, E_4\}$  such that

$$\begin{aligned} [E_1, E_2] &= E_2, & [E_1, E_3] &= -E_2 + 3E_3, & [E_1, E_4] &= 2E_4, \\ [E_2, E_3] &= [E_2, E_4] = [E_3, E_4] = 0. \end{aligned}$$

Define a left-invariant neutral metric  $g$  on  $G$  in terms of the dual basis  $\{E^i\}$  by

$$g = E^1 \otimes E^1 + E^2 \otimes E^2 - E^3 \otimes E^3 - E^4 \otimes E^4.$$

Then, it is straightforward to compute the curvature operator  $\mathcal{R}$  of  $g$  and to see that  $\tau = -12$ ,  $\mathcal{W} = 0$ ,  $\mathcal{B}^2 = 0$ , but  $\mathcal{B} \neq 0$ . More examples of such metrics are constructed in [21].

Next we consider the problem for local and global existence of holomorphic functions on hyperbolic twistor spaces. On the classical twistor space over an oriented Riemannian 4-manifold with either of the almost complex structure, there are no global non-constant holomorphic functions, even when the base manifold is non-compact [18, 19]. However, for the hyperbolic twistor spaces there is a considerable difference from the classical case as we shall see.

Recall that a  $C^\infty$  complex-valued function on an almost complex manifold is said to be *holomorphic* if its differential is complex-linear with respect to the almost complex structure. For any  $n = 0, 1, 2, 3$ , let  $\mathcal{F}_n(\mathcal{J}_i)$  denote the (possibly empty) set of points  $\sigma \in \mathcal{Z}$  such that  $n$  is the maximal number of local  $\mathcal{J}_i$ -holomorphic functions with  $\mathbb{C}$ -linearly independent differentials at  $\sigma$ . In [18] it is shown that for the classical twistor space  $\mathcal{Z}$  of an oriented Riemannian 4-manifold  $(M, g)$  we have  $\mathcal{Z} = \mathcal{F}_0(\mathcal{J}_1) \cup \mathcal{F}_3(\mathcal{J}_1) = \mathcal{F}_0(\mathcal{J}_2) \cup \mathcal{F}_1(\mathcal{J}_2)$ ; moreover,

$$\mathcal{F}_3(\mathcal{J}_1) = \pi^{-1}(\text{Int} \{p \in M : (\mathcal{W}_-)_p = 0\}),$$

$$\mathcal{F}_1(\mathcal{J}_2) = \pi^{-1}(\text{Int} \{p \in M : \mathcal{R}_p = (\mathcal{W}_+)_p\}).$$

The same arguments give this result for the hyperbolic twistor space as well.

Given a neutral almost hyperhermitian 4-manifold  $(M, g, J_1, J_2, J_3)$ , denote by  $\pi : \mathcal{Z} \rightarrow M$  the hyperbolic twistor space of  $(M, g)$ . Then, the 2-vectors corresponding to  $J_1, J_2, J_3$  via (2) form a global frame of  $\Lambda_-$  and we have a natural projection  $p : \mathcal{Z} \rightarrow H$  defined by  $p(\sigma) = (y_1, y_2, y_3)$ , where  $J_\sigma = y_1 J_1(x) + y_2 J_2(x) + y_3 J_3(x)$ ,  $x = \pi(\sigma)$ . Thus,  $\mathcal{Z}$  is diffeomorphic to  $M \times H$  by the map  $\sigma \rightarrow (\pi(\sigma), p(\sigma))$  and it is obvious that  $p$  maps any fibre of  $\mathcal{Z}$  biholomorphically on  $H$  with respect to  $\mathcal{J}_1$  and  $S$ . The conditions on  $(M, g)$ , ensuring that the natural projection  $p : \mathcal{Z} \rightarrow H$  is  $\mathcal{J}_1$ -holomorphic (resp.  $\mathcal{J}_2$ -anti-holomorphic), are the following [11]:

**Theorem 19.** *Let  $M$  be a neutral almost hyperhermitian 4-manifold and  $\mathcal{Z}$  its hyperbolic twistor space. Then, the natural projection  $p : \mathcal{Z} \rightarrow H$  is  $\mathcal{J}_1$ -holomorphic*

(resp.  $\mathcal{J}_2$ -anti-holomorphic) if and only if  $M$  is neutral hyperhermitian (resp. neutral hyperkähler).

Furthermore, we have the following result [11] in the compact case:

**Theorem 20.** *Let  $M$  be a compact neutral hyperhermitian manifold with hyperbolic twistor space  $\mathcal{Z}$  and let  $p : \mathcal{Z} \rightarrow H$  be the natural projection. Then, any  $\mathcal{J}_1$ -holomorphic function  $f$  on  $\mathcal{Z}$  has the form  $f = g \circ p$ , where  $g$  is a holomorphic function on  $H$ . If  $M$  is neutral hyperkähler, then any  $\mathcal{J}_2$ -holomorphic function  $f$  on  $\mathcal{Z}$  has the form  $f = g \circ p$  where  $g$  is an anti-holomorphic function on  $H$ .*

Now we show that in the noncompact case the situation changes radically. Recall that the Petean metrics on  $\mathbb{R}^4$  have the form:

$$g = f(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) + dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2,$$

where  $(x_1, x_2, x_3, x_4)$  are the standard coordinates on  $\mathbb{R}^4$  and  $f$  is a smooth positive function depending on  $x_1$  and  $x_2$  only. According to Theorem 10 the metrics  $g$  are neutral hyperkähler, i.e. self-dual and Ricci-flat. Hence, the almost complex structure  $\mathcal{J}_1$  on the hyperbolic twistor space  $\mathcal{Z}$  of  $(\mathbb{R}^4, g)$  is integrable. Moreover, we have the following result [11] which shows that there can be an abundance of global holomorphic functions on a hyperbolic twistor space.

**Theorem 21.** *The hyperbolic twistor space  $(\mathcal{Z}, \mathcal{J}_1)$  of  $(\mathbb{R}^4, g)$  is biholomorphic to  $\mathbb{C}^2 \times H$ .*

This result suggests the following problem: *Characterize the noncompact neutral hyperkähler 4-manifolds whose hyperbolic twistor spaces  $(\mathcal{Z}, \mathcal{J}_1)$  are Stein manifolds.*

**7.2. LeBrun-Mason twistor spaces.** As we noted in the previous subsection, the hyperbolic twistor spaces are non-compact, hence, the problem of their compactification arises. Since the fibre is a two sheeted hyperboloid, it is natural to compactify it by means of  $\mathbb{CP}^1$ . To put this simple observation in a global framework, we follow a recent twistor construction by LeBrun and Mason [32].

Denote by  $g$  the complex-bilinear extension of  $g$  to the complexification  $T^{\mathbb{C}}M$  of the tangent bundle  $TM$ . Let  $\Lambda_-^{\mathbb{C}} = \Lambda_- \otimes \mathbb{C}$  be the complexification of  $\Lambda_-$ . Set

$$\mathcal{Z} = \{[\varphi] \in \mathbb{P}(\Lambda_-^{\mathbb{C}}) : g(\varphi, \varphi) = 0\}$$

and let  $\mathcal{P} : \mathcal{Z} \rightarrow M$  be the natural projection. Then,  $\mathcal{P} : \mathcal{Z} \rightarrow M$  is a  $\mathbb{CP}^1$ -bundle. Indeed, if  $\{e_1, e_2, e_3, e_4\}$  is an oriented orthonormal frame of vector fields on an open set  $U$ , then denote by  $s_1, s_2, s_3$  the frame of  $\Lambda_-$  defined by (1). Then

$$\Psi : ([\zeta_1, \zeta_2], x) \rightarrow ([(\zeta_1^2 + \zeta_2^2)s_1(x) + (\zeta_1^2 - \zeta_2^2)s_2(x) - 2\zeta_1\zeta_2s_3(x)], x), x \in U,$$

is a biholomorphic map of  $\mathbb{CP}^1 \times U$  onto  $\mathcal{P}^{-1}(U)$ .

The bundle

$$F = \{[\varphi] \in \mathbb{P}(\Lambda_-) : g(\varphi, \varphi) = 0\}$$

is embedded in  $\mathcal{Z}$  in an obvious way. Each fibre  $F_x$  of  $F$  is sent by  $\Psi^{-1}$  onto  $\mathbb{RP}^1 \times \{x\}$ . Note also that  $\mathbb{RP}^1$  goes to the "Greenwich" meridian  $S^1 : \xi_1^2 + \xi_2^2 = 1, \xi_3 = 0$  under the standard identification  $\mathbb{CP}^1 \cong S^2$ . The manifold  $S^2 - S^1$  is biholomorphic to the disjoint union of two copies of the unit disk and the latter manifold is diffeomorphic by the stereographic projection to the two sheeted hyperboloid  $\xi_1^2 - \xi_2^2 - \xi_3^2 = 1$ . Thus  $\mathcal{Z}_x - F_x$  can be identified with the fibre of the hyperbolic twistor space of  $(M, g)$ . This can be also seen in the following way. If  $\Pi \in \mathcal{Z}_x - F_x$ , then  $\Pi \cap \overline{\Pi} = 0$ , hence,  $T_x^{\mathbb{C}}M = \Pi \oplus \overline{\Pi}$ .

Denote by  $J_x = J_x(\Pi)$  the (unique) complex structure on the vector space  $T_x M$  for which  $\Pi$  is the space of the  $(1,0)$ -vectors. The fact that the space  $\Pi$  is isotropic is equivalent to the compatibility of  $J_x$  with the metric  $g$ . If  $\{e_1, e_2 = J_x e_1, e_3, e_4 = J_x e_3\}$  is a basis of  $T_x M$  with  $\|e_1\|^2 = 1$ ,  $\|e_3\|^2 = -1$ , then  $\Pi$  is represented by the bivector  $(e_1 - ie_2) \wedge (e_3 - ie_4) = (e_1 \wedge e_3 - e_2 \wedge e_4) - i(e_1 \wedge e_4 + e_2 \wedge e_3)$ . This vector lies in  $\Lambda_-^\mathbb{C}$  since  $\Pi \in \mathbb{P}(\Lambda_-^\mathbb{C})$ , therefore, the above basis induces the orientation of  $T_x M$  which means that  $J_x$  is compatible with this orientation. Thus, the assignment  $\mathcal{Z}_x - F_x \ni \Pi \rightarrow J_x(\Pi)$  identifies  $\mathcal{Z} - F$  with the hyperbolic twistor space of  $(M, g)$ .

We have assumed that the manifold  $M$  is oriented, i.e. its structure group is reduced to  $SO(2, 2)$ . Now, suppose that the structure group of  $M$  can be reduced to the identity component  $SO_+(2, 2)$  of  $SO(2, 2)$ . In this case, following [32], we say that  $M$  is space-time orientable. This condition is equivalent to the existence of two  $g$ -orthogonal subbundles  $T_\pm$  of  $TM$  such that  $TM = T_+ \oplus T_-$  and the restriction of  $g$  to  $T_+$ , resp.  $T_-$ , is positive, resp., negative definite. To see this, choose a Riemannian metric  $h$  on  $M$ , and diagonalize  $g$  with respect to  $h$ .

Note that if the bundle  $T_\pm$  is oriented, then we can define a unique complex structure  $\mathcal{J}_\pm$  on  $T_\pm$  compatible with the metric and the orientation since  $\text{rank } T_\pm = 2$  and the restriction of  $g$  on  $T_\pm$  is a definite bilinear form: if  $e_1, e_2$  is an oriented orthonormal basis of  $T_\pm$ , then  $\mathcal{J}_\pm e_1 = e_2$ .

By definition, to fix a space-time orientation of  $M$  means to choose orientations on the bundles  $T_\pm$ . In this case  $TM$  is considered with the orientation obtained via the decomposition  $TM = T_+ \oplus T_-$ .

Suppose that  $(M, g)$  is space-time oriented and denote by  $\mathcal{J}$  the almost complex structure on  $M$  which coincides with  $\mathcal{J}_\pm$  on  $T_\pm$ . Let  $\{e_1, e_2 = \mathcal{J}e_1\}$  and  $\{e_3, e_4 = \mathcal{J}e_3\}$  be oriented orthonormal bases of  $(T_+)_x$  and  $(T_-)_x$ , respectively,  $x \in M$ . Then, the 2-plane of  $(1,0)$ -vectors of  $\mathcal{J}_x$  is represented by the bivector  $(e_1 - ie_2) \wedge (e_3 - ie_4) = s_2 - is_3$ , while the 2-plane corresponding to the conjugate complex structure  $-\mathcal{J}_x$  is represented by  $s_2 + is_3$ . The images of  $[s_2 - is_3]$  and  $[s_2 + is_3]$  under  $\Psi^{-1}$  are the points  $[1, i]$  and  $[1, -i]$  in  $\mathbb{CP}^1$  which go to  $(0, 1, 0)$  and  $(0, -1, 0)$  under the standard identification  $\mathbb{CP}^1 \cong S^2$ . The latter points determine the two connected components of  $S^2 - S^1$ . Thus, if  $(M, g)$  is space-time oriented, then the space  $\mathcal{Z} - F$  has two connected components – one of them contains the almost complex structure  $\mathcal{J}$  and the other one contains the conjugate almost complex structure  $-\mathcal{J}$ . Let  $U$  be the component of  $\mathcal{Z} - F$  determined by  $\mathcal{J}$ . Then, the closure  $\mathcal{Z}_+ = U \cup F$  of  $U$  in  $\mathcal{Z}$  is a compact 6-dimensional manifold with boundary.

Now, suppose that the image of every maximally extended null geodesic in  $M$  is an embedded circle in  $M$ . A manifold with this property is called Zollfrei and we refer to [32] for more information about these manifolds. It is proved in [32] that if  $(M, g)$  is a space-time oriented self-dual Zollfrei 4-manifold, then the space  $F$  is a trivial 2-sphere bundle  $q : F \rightarrow P$  over a manifold  $P$  diffeomorphic to  $\mathbb{RP}^3$ . Let us endow the disjoint union  $Z = U \amalg P$  with the quotient topology induced by the map  $\mathcal{Z}_+ \rightarrow Z$  which is the identity map on  $U$  and the map  $q$  on  $F$ . Then,  $Z$  is a compact topological 6-manifold. One of the key results in [32] is that  $Z$  admits a complex structure which coincides with the Atiyah-Hitchin-Singer structure  $\mathcal{J}_1$  on  $U$ , identified with a connected component of the hyperbolic twistor space of  $M$ . The manifold  $Z$  is called the twistor space of  $(M, g)$ .

If  $(M, g)$  is an oriented self-dual Zollfrei manifold which is not space-time orientable, then it possesses a double cover  $(\widetilde{M}, \widetilde{g})$  which is space-time orientable, self-dual and

Zollfrei. The twistor space of  $(\widetilde{M}, \widetilde{g})$  is called the twistor space of  $(M, g)$ .

It is proved in [32] that if  $(M, g)$  is a self-dual Zollfrei 4-manifold and if  $M$  is space-time orientable, then it is homeomorphic to  $S^2 \times S^2$ ; if  $M$  is not space-time orientable, then it is homeomorphic to the quadric

$$\mathbb{M}^{2,2} = \{[x, y] \in \mathbb{RP}^5 : |x|^2 - |y|^2 = 0\},$$

which is the quotient of  $S^2 \times S^2$  by the  $\mathbb{Z}_2$ -action generated by the map  $(x, y) \rightarrow (-x, -y)$ . It is also shown in [32] that the twistor space of either of  $S^2 \times S^2$  or  $\mathbb{M}^{2,2}$  with the natural neutral metric is biholomorphic to  $\mathbb{CP}^3$  in such a way that  $P \subset Z$  becomes the standard  $\mathbb{RP}^3 \subset \mathbb{CP}^3$ . Hence, the hyperbolic twistor space of  $S^2 \times S^2$  is  $\mathbb{CP}^3 - \mathbb{RP}^3$ . It is also a result of [32] that the twistor space of any self-dual Zollfrei manifold is diffeomorphic to  $\mathbb{CP}^3$  in such a way that the Chern classes of  $Z$  are sent to the Chern classes of  $\mathbb{CP}^3$ .

Finally, let us note that the LeBrun-Mason twistor construction can be reversed to produce self-dual metrics of neutral signature. We refer to [32] for details.

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## ГЕОМЕТРИЯ НА НЕУТРАЛНИ МЕТРИКИ В РАЗМЕРНОСТ ЧЕТИРИ

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Настоящата статия е обзор на някои скорошни резултати върху геометрията на неутралните метрики в размерност четири и техните туисторни пространства. Разглеждат се следните теми: Неутрални Келерови и хиперкелерови повърхнини, Метрики на Уокър, Неутрални анти-автодуални 4-многообразия и проективни структури, Туисторни пространства на неутрални метрики.