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ONE CONNECTION ON A LOCALLY DECOMPOSABLE RIEMANNIAN SPACE*

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Let a Riemannian space M with a metric g admit an almost product structure J, which preserves the scalar product. By mean of the Riemannian connection of g we define another affine connection on M, and we find a subclass of the locally decomposable almost Einstein spaces with constant totally real curvature.

1. Introduction. We consider a Riemannian space $M(\dim M = n)$ with a metric g admitting an almost product structure J, which preserves the scalar product, i.e.

$$J^2 = id\ (J \neq id), \quad g(Jx, Jy) = g(x, y), \quad x, y \in \chi M. \tag{1}$$

Let ∇ be the Riemannian connection of g. If

$$\nabla J = 0, \tag{2}$$

then M is a locally decomposable Riemannian space [2], [3], i.e. $M = M_1 \times M_2$, where both M_1, M_2 are Riemannian spaces. If $\dim M_1 = p$ and $\dim M_2 = q$, then $n = p + q$.

2. The Connection $\overline{\nabla}$. Now let M be a locally decomposable Riemannian space, and Γ_{ij}^k be the Chrisoffel symbols of ∇. Analogously to the case of B-manifold [1], we define another affine connection $\overline{\nabla}$ by the condition

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + T_{ij}^k,\ T_{ij}^k = -g_{ij} f^k - J_{ij} \tilde{f}^k, \tag{3}$$

where f is a smooth vector field on M, $J_{ij} = J_i^p g_{pj}$ and $\tilde{f}^j = J_j^p f^p$.

Theorem 2.1. *Let M be a locally decomposable Riemannian space, and let ∇ and $\overline{\nabla}$ satisfy* (3). *Then, $\overline{\nabla}$ is a symmetric affine connection and*

$$\overline{\nabla} J = 0. \tag{4}$$

Proof. The conditions (3) imply that $\overline{\nabla}$ is a symmetric connection. By straightforward calculations from (3) we get $\overline{\nabla}_i J_j^k = \nabla_i J_j^k$. By using (2), we prove (4). So that J is a covariant constant. □

We note that $J_{ij} = J_{ji}$, and from (2), (3) we find:

$$\overline{\nabla}_i g_{jk} = f_j g_{ik} + f_k g_{ij} + \tilde{f}_j J_{ik} + \tilde{f}_k J_{ij},$$

$$\overline{\nabla}_i J_{jk} = f_j J_{ik} + f_k J_{ij} + \tilde{f}_j g_{ik} + \tilde{f}_k g_{ij},$$

i.e. $\overline{\nabla}$ is not a metric connection.

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For the curvature tensor fields $\overline{R}$ of $\overline{\nabla}$ and R of ∇, it is well known the following identity:

$$\overline{R}^h_{ijk} = R^h_{ijk} + \nabla_j T^h_{ik} - \nabla_k T^h_{ij} + T^s_{ik} T^h_{sj} - T^s_{ij} T^h_{sk}. \tag{5}$$

From (3) and (5) we obtain

$$\overline{R}^h_{ijk} = R^h_{ijk} - J_{ik}\tilde{P}^h_j + J_{ij}\tilde{P}^h_k - g_{ik}P^h_j + g_{ij}P^h_k, \tag{6}$$

where

$$P^h_j = \nabla_j f^h - f_j f^h - \tilde{f}_j \tilde{f}^h \tag{7}$$

and $\tilde{P}^h_j = J^h_s P^s_j$. From (2) we have

$$\tilde{P}^h_j = \nabla_j \tilde{f}^h - f_j \tilde{f}^h - \tilde{f}_j f^h. \tag{8}$$

Theorem 2.2. *Let M be a locally decomposable Riemannian space, and let $\overline{\nabla}$ and ∇ satisfy* (3)*. Then, $\overline{\nabla}$ is an equiaffine connection if and only if the vector field f is gradient.*

Proof. We know that ∇ is an equiaffine connection, so $R^i_{ijk} = 0$. Then, contracting (6) by $h = i$, we get

$$\overline{R}^i_{ijk} = 2(P_{kj} - P_{jk}), \quad P_{kj} = P^s_k g_{sj}. \tag{9}$$

From (9) we conclude that $\overline{\nabla}$ is equiaffine if and only if $P_{jk} = P_{kj}$. Due to (7) the last condition is satisfied, if and only if, $\nabla_k f_j = \nabla_j f_k$. The last equality is a necessary and sufficient condition for f to be gradient. The theorem is proved. □

Theorem 2.3. *Let M be locally decomposable Riemannian space, and let ∇ be the Riemannian connection of g. If f is a smooth vector field on M, and $\overline{\nabla}$ is a locally flat connection, defined by the relation* (3)*, then the curvature tensor field R of M satisfies the identity:*

$$\begin{aligned} R(x,y,z,u) = \frac{1}{A}[&(2(1-n)(trJ)\tau^* + (n(n-2) + tr^2 J)\tau)(g(x,u)g(y,z) \\ &- g(y,u)g(x,z) + g(Jx,u)g(Jy,z) - g(Jy,u)g(Jz,x)) \\ &+ (2(1-n)(trJ)\tau + (n(n-2) + tr^2 J)\tau^*)(g(Jx,u)g(y,z) \\ &- g(y,u)g(Jx,z) + g(x,u)g(Jy,z) - g(Jy,u)g(z,x))], \end{aligned} \tag{10}$$

where $x,y,z,u \in \chi(M)$, $n-2 \neq \pm trJ$, $A = (n^2 - tr^2 J)((n-2)^2 - tr^2 J)$. Moreover, M is an almost Einstein space.

Proof. We consider a locally decomposable Riemannian space M, i.e. $\nabla J = 0$. We assume that $\overline{\nabla}$ is a locally flat connection, for which it is necessary and sufficient that $\overline{R} = 0$. From (6) we find:

$$R^h_{ijk} = J_{ik}\tilde{P}^h_j - J_{ij}\tilde{P}^h_k + g_{ik}P^h_j - g_{ij}P^h_k, \tag{11}$$

and from $R^i_{ijk} = 0$ we get $P_{kj} = P_{jk}$. So f is a gradient vector field.

With the help of the identity $R_{hijk} = R_{jkhi}$ and the equation (11), we find:

$$J_{ik}(\tilde{P}_{jh} - \tilde{P}_{hj}) - J_{ij}\tilde{P}_{kh} + J_{kh}\tilde{P}_{ij} + g_{kh}P_{ij} - g_{ij}P_{kh} = 0, \quad \tilde{P}_{jh} = \tilde{P}^s_j g_{sh}.$$

In the last equation we contract with J^{ik}, and obtain:

$$\tilde{P}_{jh} = \tilde{P}_{hj}. \tag{12}$$

The equations (8), (12) imply that $\tilde{f}$ is also a gradient vector field.

By using the equation (11), we find the Ricci tensor field $S_{ij} = R^k_{ijk}$ as follows:

$$S_{ij} = -\tilde{P}J_{ij} + 2P_{ij} - Pg_{ij}, \quad P = P^k_k, \tilde{P} = \tilde{P}^k_k. \tag{13}$$

For the first scalar curvature $\tau = S_{ij}g^{ij}$ we get

$$\tau = (2-n)P - (\mathrm{tr}J)\tilde{P}. \tag{14}$$

On the other hand, contracting (11) with g^{ij} and using (12), we have

$$S_{kh} = (2-n)P_{kh} - (\mathrm{tr}J)\tilde{P}_{kh}. \tag{15}$$

For the second scalar curvature $\tau^* = S_{ij}J^{ij}$ we get

$$\tau^* = (2-n)\tilde{P} - (\mathrm{tr}J)P. \tag{16}$$

If we denote $\tilde{S}_{ij} = S_{ip}J^p_j$, then from (15) we obtain

$$\tilde{S}_{ij} = (2-n)\tilde{P}_{ij} - (\mathrm{tr}J)P_{ij}. \tag{17}$$

With the help of the identities (13), (15), and (17) we find

$$\begin{aligned} S_{ij} = \frac{1}{n^2 - \mathrm{tr}^2 J}&[((n(2-n) + \mathrm{tr}^2 J)P + ((n-2)\mathrm{tr}J - n\mathrm{tr}J)\tilde{P})g_{ij} \\ &+ ((n(2-n) + \mathrm{tr}^2 J)\tilde{P} + ((n-2)\mathrm{tr}J - n\mathrm{tr}J)P)J_{ij}]. \end{aligned}$$

From (14), (16), and the last equation we obtain

$$S_{ij} = \frac{1}{n^2 - \mathrm{tr}^2 J}((n\tau - (\mathrm{tr}J)\tau^*)g_{ij} + (n\tau^* - (\mathrm{tr}J)\tau)J_{ij}). \tag{18}$$

So we prove that M is an almost Einstein space.

From (11), (15), (17), and (18) we get (10) in finally local coordinates. □

Corollary 2.4. *If a connected space M satisfies the conditions of Theorem* 2.3*, then M is a space with constant totally real curvature.*

Proof. For the totally real section $\sigma = \{x, y\}$ in T_pM, $p \in M$, we have that $g(x, Jx) = g(y, Jy) = g(x, Jy) = 0$ and the sectional curvature μ of σ is

$$\mu(\sigma) = \frac{R(x,y,x,y)}{g(x,x)g(y,y) - g^2(x,y)}.$$

Then by using (10) we get

$$\mu(\sigma) = \frac{2(1-n)(\mathrm{tr}J)\tau^* + (n(n-2) + \mathrm{tr}^2 J)\tau))}{(\mathrm{tr}^2 J - n^2)((n-2)^2 - \mathrm{tr}^2 J}.$$

On the other hand we have $\nabla_i S^i_k = \dfrac{1}{2}\nabla_k \tau$, $\nabla_i \tilde{S}^i_k = \frac{1}{2}\nabla_k \tau^*$. From (18) and the last identities we obtain the system

$$\begin{aligned} (\mathrm{tr}^2 J + 2n - n^2)\tau_i - (\mathrm{tr}J)\tau^*_i + n\tilde{\tau}^*_i - (\mathrm{tr}J)\tilde{\tau}_i = 0 \\ -(\mathrm{tr}J)\tau_i + n\tau^*_i - (\mathrm{tr}J)\tilde{\tau}^*_i + (\mathrm{tr}^2 J + 2n - n^2)\tilde{\tau}_i = 0 \\ -(\mathrm{tr}J)\tau_i + (\mathrm{tr}^2 J + 2n - n^2)\tau^*_i - (\mathrm{tr}J)\tilde{\tau}^*_i + n\tilde{\tau}_i = 0 \\ n\tau_i - (\mathrm{tr}J)\tau^*_i + (\mathrm{tr}^2 J + 2n - n^2)\tilde{\tau}^*_i - (\mathrm{tr}J)\tilde{\tau}_i = 0, \end{aligned}$$

where $\tau_i = \nabla_i \tau$, $\tau^*_i = \nabla_i \tau^*$, $\tilde{\tau}_i = J^s_i \tau_s$, $\tilde{\tau}^*_i = J^s_i \tau^*_s$. The determinant of this system is $D = (n^2 - n - \mathrm{tr}^2 J)^2(n^2 - 3n - \mathrm{tr}^2 J - 2\mathrm{tr}J)(n^2 - 3n - \mathrm{tr}^2 J + 2\mathrm{tr}J)$. We have that $D \neq 0$, and then the only solution is $\tau_i = \tau^*_i = \tilde{\tau}_i = \tilde{\tau}^*_i = 0$, which implies $\tau = \tau^* =$const. So μ

is a constant too. $\square$

Corollary 2.5. *Let M satisfy the conditions of Theorem 2.3 and $trJ = 0$. Then the curvature tensor field R of M has the form:*

$$R(x,y,z,u) = \frac{\tau}{n(n-2)}((g(x,u)g(y,z) - g(y,u)g(x,z) + g(Jx,u)g(Jy,z) - g(Jy,u)g(Jz,x)) + \frac{\tau^*}{n(n-2)}(g(Jx,u)g(y,z) - g(y,u)g(Jx,z) + g(x,u)g(Jy,z) - g(Jy,u)g(z,x)), \quad n-2 \neq 0.$$

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ЕДНА СВЪРЗАНОСТ ВЪРХУ ПРОСТРАНСТВО НА ЛОКАЛНА ДЕКОМПОЗИЦИЯ

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Нека M е римановo пространство с метрика g, допускащо допълнителна структура на почти произведение J, която запазва скаларното произведение. С помощта на римановата свързаност ∇, породена от метриката g, дефинираме друга афинна свързаност и получаваме подклас на почти айнщайнови пространства с постоянна напълно реална секционна кривина.