МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2008
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2008
Proceedings of the Thirty Seventh Spring Conference of
the Union of Bulgarian Mathematicians
Borovetz, April 2–6, 2008

NOTES ON REGULAR ORBITS UNDER FAMILIES OF OPERATORS*

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In this note is considered the problem of the existence of a dense set of vectors in a Banach space whose orbits under a family of operators tend strongly to infinity.

1. Introduction. Throughout this paper, $\mathcal{X}$ denote an infinite dimensional complex Banach space and $\mathcal{B}(\mathcal{X})$ is the algebra of all bounded linear operators on $\mathcal{X}$. For $T \in \mathcal{B}(\mathcal{X})$ by $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$ and $r(T)$ we denote the spectrum, the approximate point spectrum, the point spectrum and the spectral radius of T, respectively.

An *orbit* of $x \in \mathcal{X}$ under $T \in \mathcal{B}(\mathcal{X})$ is the sequence

$$\mathrm{Orb}(T,x) := \{x, Tx, T^2x, \ldots\}.$$

The orbits under a single operator can behave quite differently. An operator can have some orbits very regular (i.e. orbits tending strongly to 0 or tending strongly to infinity) and other orbits extremely irregular (like the one of a hypercyclic vector, i.e. a vector whose orbit is dense in the whole space). For an example of an operator for which the space contains a dense set of vectors with orbits tending strongly to infinity, a dense set of hypercyclic vectors and even a dense set of vectors with orbits tending strongly to 0, we refer the reader to [1, Ch.III Sec.1.C].

In this paper we study regular orbits tending strongly to infinity: $\|T^n x\| \to +\infty$, as $n \to +\infty$. Although the behavior of an orbit depends strongly on the initial vector $x \in \mathcal{X}$, the existence of orbits tending strongly to infinity is closely related to the spectrum of the operator. More precisely, by Corrolary 1.5 in [8] and the Spectral Mapping Theorem, we have

Theorem 1.1. *If $T \in \mathcal{B}(\mathcal{X})$, $\lambda \in \sigma(T)$ and $(\alpha_n)_{n\geq 1}$ is a sequence of positive numbers such that $\sum_{n\geq 1} \alpha_n^{2/3} < +\infty$, then for every $x \in \mathcal{X}$ and every $\varepsilon > 0$ there is a positive integer $n_0 = n_0(\varepsilon)$ and a vector $z \in \mathcal{X}$ satisfying $\|x - z\| < \varepsilon$ and*

$$\|T^n z\| \geq \alpha_n |\lambda|^n, \text{ for all } n \geq n_0.$$

Corollary 1.2. *If $T \in \mathcal{B}(\mathcal{X})$ is with $r(T) > 1$, then there is a dense set $D \subset \mathcal{X}$ such that $\mathrm{Orb}(T,x)$ tends strongly to infinity for every $x \in D$.*

*2000 Mathematics Subject Classification: Primary 47A05, secondary 47A25, 47A60.
Key words: Orbits tending to infinity, spectrum, Banach space, families of operators.

Before continuing, let us mention that in the settings of reflexive Banach spaces and, in particular, in the case of operators on a Hilbert spaces, as the proofs of the Beauzamy's results in [1, Theorem III.2.A.1] and [1, Theorem III.2.A.5] suggest (for a complete proof of the second result see [2]), for $\lambda \in \sigma_{ap}(T)\backslash\sigma_p(T)$ we can give better estimates than the one in Theorem 1.1 as follows.

Theorem 1.3. *Let $\mathcal{X}$ be a reflexive Banach space, $T \in \mathcal{B}(\mathcal{X})$, $(\alpha_n)_{n\geq 1}$ is a sequence of positive numbers and $\lambda \in \sigma_{ap}(T)\backslash\sigma_p(T)$.*

(a) If $\sum_{n\geq 1} \alpha_n < +\infty$, then every open ball in $\mathcal{X}$ with radius strictly larger than $\sum_{n\geq 1} \alpha_n$ contains a vector $z \in \mathcal{X}$ satisfying

$$\|T^n z\| \geq \alpha_n |\lambda|^n/2, \text{ for all } n \geq 1.$$

(b) If $\mathcal{X}$ is a Hilbert space and $(\alpha_n)_{n\geq 1}$ strictly deceases to 0, then every open ball in $\mathcal{X}$ with radius strictly larger than α_1 contains a vector $z \in \mathcal{X}$ satisfying

$$\|T^n z\| \geq \alpha_n |\lambda|^n, \text{ for all } n \geq 1.$$

2. Orbits under families of operators. The main results in this section are based on V. Müller's results stated as Lemma 1.4 and Corollary 1.5 in [8]. For our purpose we are going to use the following modification of Lemma 1.4:

Lemma 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\{T_\alpha : \alpha \in I\}$ be a family of operators on $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, $(a_\alpha)_{\alpha\in I}$ a family of positive numbers such that $\sum_{\alpha\in I} a_\alpha^{2/3} < 1/4$ and $x \in \mathcal{X}$. Then, there exists $u \in \mathcal{X}$ such that $\|x - u\| \leq 1/4$ and*

$$\|T_\alpha u\| \geq a_\alpha \|T_\alpha\|, \text{ for all } \alpha \in I.$$

Proof. Let $\delta > 0$ be such that $(1 + \delta) \sum_{\alpha\in I} a_\alpha^{2/3} < 1/4$. For $\alpha \in I$, let $z_\alpha \in \mathcal{X}$ be such that $\|z_\alpha\| = 1$ and $\|T_\alpha z_\alpha\| \geq (1 + \delta)^{-1}\|T_\alpha\|$, and put $\varepsilon_\alpha = (1 + \delta)a_\alpha^{2/3}$. Let

$$\mathcal{F} = \{F \subset I : F \text{ is finite}\}.$$

Claim 2.1.A. For every $F \in \mathcal{F}$ there exists a set $\{\lambda_\alpha : \alpha \in F\} \subset \mathbb{C}$ such that $|\lambda_\alpha| \leq \varepsilon_\alpha$ for all $\alpha \in F$ and

$$\left\| T_\beta \left(x + \sum_{\alpha\in F} \lambda_\alpha z_\alpha \right) \right\| \geq a_\beta \|T_\beta\|, \text{ for all } \beta \in F. \tag{2.1}$$

Proof of Claim 2.1.A. The same as the proof of part A in [8, Lemma 1.4].

Claim 2.1.B. The set $M = \left\{x + \sum_{\alpha\in I} \lambda_\alpha z_\alpha : |\lambda_\alpha| \leq \varepsilon_\alpha,\ \alpha \in I\right\}$ is totally bounded.

Proof of Claim 2.1.B. Let $\varepsilon' > 0$ and $F_0 \in \mathcal{F}$ be such that $\sum_{\alpha\in I\backslash F_0} \varepsilon_\alpha < \varepsilon'/2$. Then, the set

$$M_{F_0} = \left\{ x + \sum_{\alpha\in F_0} \lambda_\alpha z_\alpha : |\lambda_\alpha| \leq \varepsilon_\alpha,\ \alpha \in F_0 \right\}$$

is compact, therefore, we can find $y_1, \ldots, y_n \in M_{F_0}$ such that

$$M_{F_0} \subseteq B(y_1, \varepsilon'/2) \cup \cdots \cup B(y_n, \varepsilon'/2). \tag{2.2}$$

Let $y = x + \sum_{\alpha \in I} \lambda'_\alpha z_\alpha \in M$. Then,

$$y = x + \sum_{\alpha \in F_0} \lambda'_\alpha z_\alpha + \sum_{\alpha \in I \setminus F_0} \lambda'_\alpha z_\alpha = y' + \sum_{\alpha \in I \setminus F_0} \lambda'_\alpha z_\alpha$$

where $y' = x + \sum_{\alpha \in F_0} \lambda'_\alpha z_\alpha \in M_{F_0}$. By (2.2) there exists $1 \leq k \leq n$ such that $\|y' - y_k\| < \varepsilon'/2$. Then

$$\|y - y_k\| = \left\| y' - y_k + \sum_{\alpha \in I \setminus F_0} \lambda'_\alpha z_\alpha \right\| \leq \|y' - y_k\| + \sum_{\alpha \in I \setminus F_0} \varepsilon_\alpha < \varepsilon'.$$

Proof of the Lemma 2.1 – continued. By Claim 2.1.A., for every $F \in \mathcal{F}$ there exists a vector $u_F \in M_F \subset M$ such that

$$\|T_\alpha u_F\| \geq a_\alpha \|T_\alpha\|, \text{ for all } \alpha \in F. \tag{2.3}$$

Thus, we obtain a net $(u_F)_{F \in \mathcal{F}}$ in M. By Claim 2.1.B., $(u_F)_{F \in \mathcal{F}}$ contains a convergent subnet, i.e. there exist a directed set B, a subfamily $\{F_\beta : \beta \in B\} \subset \mathcal{F}$ and $u \in \mathcal{X}$ such that:

1. for every $F_0 \in \mathcal{F}$ there is $\beta_0 \in B$ so that $F_\beta \supseteq F_0$, for all $\beta \geq \beta_0$;
2. $(u_{F_\beta})_{\beta \in B}$ converges to u.

Then,

a. $\|x - u\| \leq \|x - u_{F_\beta}\| + \|u_{F_\beta} - u\| < 1/4 + \|u_{F_\beta} - u\|$, for every $\beta \in B$, and consequently $\|x - u\| \leq 1/4$;

b. if $\alpha_0 \in I$ is fixed and $\beta_0 \in B$ is such that $F_\beta \supset \{\alpha_0\}$ for every $\beta \geq \beta_0$, then by (2.3), $\|T_{\alpha_0} u_{F_\beta}\| \geq a_{\alpha_0} \|T_{\alpha_0}\|$ and, consequently,
$$\|T_{\alpha_0} u\| = \lim_{\beta \in B} \|T_{\alpha_0} u_{F_\beta}\| \geq a_{\alpha_0} \|T_{\alpha_0}\|. \qquad \square$$

Corollary 2.2. *Let $\{T_\alpha : \alpha \in I\}$ be an arbitrary family of operators in $\mathcal{B}(\mathcal{X})$, $\{(a_{\alpha,n})_{n \in \mathbb{N}} : \alpha \in I\}$ be a family of sequences of positive numbers such that $\sum_{(\alpha,n) \in I \times \mathbb{N}} a_{\alpha,n}^{2/3} < +\infty$ and $x \in \mathcal{X}$. Then, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ and $z \in \mathcal{X}$ satisfying $\|x - z\| < \varepsilon$ and*

$$\|T_\alpha^n z\| \geq a_{\alpha,n} \|T_\alpha^n\|, \textit{ for every } \alpha \in I \textit{ and } n \geq n_0.$$

Proof. Let $\mathcal{G} = \{G \subset I \times \mathbb{N} : G \text{ is finite}\}$. Since $\sum_{(\alpha,n) \in I \times \mathbb{N}} a_{\alpha,n}^{2/3} < +\infty$, there are $G_0 \in \mathcal{G}$ and $a > 0$ such that

$$\left(8 \sum_{(\alpha,n) \in (I \times \mathbb{N}) \setminus G_0} a_{\alpha,n}^{2/3} \right)^{3/2} < a < \varepsilon.$$

Let $m_0 = \max\{n \in \mathbb{N} : \exists \alpha \in I,\ (\alpha, n) \in G_0\}$ and let k_0 be the number of elements of G_0. We define

$$b_{\alpha,n} = \begin{cases} (8k_0)^{-3/2}, & \text{if } (\alpha, n) \in G_0 \\ a^{-1} a_{\alpha,n}\ , & \text{if } (\alpha, n) \in (I \times \mathbb{N}) \setminus G_0. \end{cases}$$

Then

$$\sum_{(\alpha,n)\in I\times\mathbb{N}} b_{\alpha,n}^{2/3} = \sum_{(\alpha,n)\in(I\times\mathbb{N})\setminus G_0} b_{\alpha,n}^{2/3} + \sum_{(\alpha,n)\in G_0} b_{\alpha,n}^{2/3} < \frac{1}{8} + \sum_{(\alpha,n)\in G_0} \frac{1}{8k_0} = \frac{1}{4}.$$

Applying Lemma 2.1 on the family $\{T_{\alpha,n} = T_\alpha^n : (\alpha, n) \in I \times \mathbb{N}\}$, we obtain that there is $u \in \mathcal{X}$ such that

$$\|a^{-1}x - u\| \le 1/4 \text{ and } \|T_\alpha^n u\| \ge b_{\alpha,n}\|T_\alpha^n\|, \text{ for all } (\alpha, n) \in I \times \mathbb{N}.$$

Then, the vector $z = au$ will satisfy

$$\|x - z\| \le a/4 < \varepsilon \text{ and } \|T_\alpha^n z\| \ge ab_{\alpha,n}\|T_\alpha^n\|, \text{ for all } (\alpha, n) \in I \times \mathbb{N}.$$

In particular, if $n \ge n_0 = m_0 + 1$, then from the definition of m_0 we have $(\alpha, n) \notin G_0$, and, consequently,

$$\|T_\alpha^n z\| \ge a_{\alpha,n}\|T_\alpha^n\|, \text{ for all } \alpha \in I \text{ and } n \ge n_0. \qquad \square$$

3. Remarks on orbits tending strongly to infinity. In view of Corollary 1.2 and the results in Section 2, it seems natural to propose the following

Question 3.1. *If $(T_\alpha)_{\alpha\in I}$ is a family in $\mathcal{B}(\mathcal{X})$ and $r(T_\alpha) > 1$, for all $\alpha \in I$, will then $\mathcal{X}$ contain a dense set D such that $\mathrm{Orb}(T_\alpha, x)$ tends strongly to infinity for every $x \in D$ and $\alpha \in I$?*

We do not know the answer of this question. However, we have an affirmative answer to special cases. Extending Theorem 1.3 (a) and (b) for pairs of operators, in [3] and [4] we have proved the existence of a dense set of vectors with orbits tending strongly to infinity under pair of operators T and S on Hilbert or reflexive Banach spaces, when both $\sigma_{ap}(T)\setminus\sigma_p(T)$ and $\sigma_{ap}(S)\setminus\sigma_p(S)$ have a nonempty intersection with $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$. This result can be easily extended up to a finite family of operators. For a sequence of operators $(T_i)_{i\ge1}$ on Hilbert spaces or reflexive Banach spaces we had to make some additional restrictions. In [5] and [6] we have proved the existence of a dense set of vectors D such that $\mathrm{Orb}(T_i, x)$ tends strongly to infinity for every $x \in D$ and $i \ge 1$, when there is $\beta > 0$ so that $\sigma_{ap}(T_i)\setminus\sigma_p(T_i)$ has a nonempty intersection with $\{\lambda \in \mathbb{C} : |\lambda| > 1 + \beta\}$, for every $i \ge 1$.

Question 3.1 can be answered affirmatively for a finite family of operators $\{T_\alpha : \alpha \in I\}$ on arbitrary Banach space: as in [3] and [4], we can always make an appropriate choice of the sequences $\{(a_{\alpha,n})_{n\in\mathbb{N}} : \alpha \in I\}$, and apply Corollary 2.2 and the Spectral Mapping Theorem. As in the case of sequences of operators on Hilbert or reflexive Banach spaces, we can give a partial answer to Question 3.1.

Corollary 3.2. *If $(T_\alpha)_{\alpha\in I}$ is a countable family of operators in $\mathcal{B}(\mathcal{X})$ and there is $\beta > 0$ so that $r(T_\alpha) > 1 + \beta$, for all $\alpha \in I$, then $\mathcal{X}$ contains a dense set D such that $\mathrm{Orb}(T_\alpha, x)$ tends strongly to infinity for every $x \in D$ and $\alpha \in I$.*

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БЕЛЕЖКИ ВЪРХУ РЕГУЛЯРНИ ОРБИТИ НА ФАМИЛИЯ ОТ ОПЕРАТОРИ

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Разглежда се проблемът за съществуване на гъсто множество от вектори в Банахово пространство, чиито орбити относно действието на фамилия от оператори клонят строго към безкрайност.