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TRANSITIVE OPERATORS ON $\mathbb{R}^{\mathbb{R}}$*

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Let X be a topological space and $Y \subset X$. The map $f : Y \to X$ is said to be *transitive* if there exists an element $t \in Y$ for which the forward orbit

$$O_+f(t) = \{f^n(t)|n \in N\}$$

is a dense subset of X. In this paper we consider an example of a transitive map in the non countable product $X = \mathbb{R}^{\mathbb{R}}$ (recall that X has an uncountable weight).

1. Let X be a topological space and $Y \subset X$. The map $f : Y \to X$ is said to be *transitive* if there exists an element $t \in Y$ for which the forward orbit $O_+f(t) = \{f^n(t)|n = 1, 2, \cdots\}$ is a dense subset of X. We call such a point t, as the custom is, a *hypercyclic element* of f [4].

This paper contains a sufficient transitivity condition for generalized shift operators on the uncountable product $\mathbb{R}^{\mathbb{R}} = \prod\limits_{\alpha\in\mathbb{R}} \mathbb{R}_\alpha$ where for every α, $\mathbb{R}_\alpha$ is a copy of the real line $\mathbb{R}$. Note that the space $\mathbb{R}^{\mathbb{R}}$ is separable [1]. Here, occasionally, we use some thoughts from [2].

It is a folklore fact that transitivity is a topological property:

Suppose that $\mathbf{f} : X \to X$ *is transitive and let* $g : X \to Y$ *be a homeomorphism. Then,* $\mathbf{h} = g \circ \mathbf{f} \circ g^{-1}$ *is also transitive.*

To define a generalized shift operator, we need the notion of the *shift map*:

The injective map $\nu : \mathbb{N} \to \mathbb{N}$ of the positive integers is said to be a shift map when the following condition holds:

$$\bigcap_{n=1}^{\infty} \nu^n(\mathbb{N}) = \emptyset.$$

2. Further let X_n be a separable topological space for every integer n, and let $g_n : X_{\nu(n)} \to X_n$ be a surjective map. Next, denote by $\mathbb{X}$ the product $\prod\limits_{n=1}^{\infty} X_n$.

Definition 2.1. *The map* $\mathbf{g} : \mathbb{X} \to \mathbb{X}$ *defined by the formula*

$$\mathbf{g}_\nu(\mathbf{x}) = \mathbf{g}(\mathbf{x}) = (g_1(x(\nu(1))), g_2(x(\nu(2))), \dots, g_n(x(\nu(n))), \cdots)$$

is called a generalized shift operator.

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Denote now for an integer $n \in \mathbb{N}$ by $\mathbb{I}(n)$ the set $\mathbb{I}(n) = \{1, 2, \ldots, n\}$. To prove the transitivity of $\mathbf{g}$ we need the following two lemmas.

Lemma 2.2. *For every k there exists an integer $n_k \in \mathbb{N}$ such that for $n \geq n_k$ we have $\mathbb{I}(k) \cap \nu^n(\mathbf{N}) = \emptyset$.*

Proof. We have $\emptyset = \bigcap_{n=1}^{\infty} \nu^n(\mathbb{N})$. The map ν is injective, hence, $\emptyset = \mathbb{I}(k) \cap \bigcap_{n=1}^{\infty} \nu^n(\mathbb{N}) = \bigcap_{n=1}^{\infty} (\nu^n(\mathbb{N}) \cap \mathbb{I}(k))$. Since the set $\mathbb{I}(k)$ is finite, it follows from the above that $\nu^{n_k}(\mathbb{N}) \cap \mathbb{I}(k) = \emptyset$ for some $n_k \in \mathbb{N}$. From the inclusion $\nu^{n+1}(\mathbb{N}) \subset \nu^n(\mathbb{N})$ one obtains that $\mathbb{I}(k) \cap \nu^n(\mathbb{N}) = \emptyset$ for $n \geq n_k$. Evidently, one may choose in addition the sequence $\{n_k\}$ to be increasing.

Lemma 2.3. *If ν is a shift function, then there exists a sequence $p_1 < p_2 < \cdots$ of integers such that if $k \neq l$, then $\nu^{p_k}(\mathbb{I}(k)) \cap \nu^{p_l}(\mathbb{I}(l)) = \emptyset$.*

Proof. We may put for example $p_k = n_1 + \cdots + n_k$. Thus, for $k < l$ we have $\emptyset = \mathbb{I}(k) \cap \nu^{n_k}(\mathbb{N}) = \nu^{p_k}(\mathbb{I}(k) \cap \nu^{n_k}(\mathbb{N}))$. Thus, for the injective function ν^{p_k} we obtain $\nu^{p_k}(\mathbb{I}(k)) \cap \nu^{p_k+n_k}(\mathbb{N}) = \emptyset$. Evidently $p_l \geq p_k + n_k$ and $\nu^{p_l}(\mathbb{I}(l)) \subset \nu^{p_l}(\mathbb{N}) \subset \nu^{p_k+n_k}(\mathbb{N})$, so $\nu^{p_k}(\mathbb{I}(k)) \cap \nu^{p_l}(\mathbb{I}(l)) = \emptyset$.

Now we can prove the transitivity of the shift operator.

Theorem 2.4. *The operator* $\mathbf{g}$ *is transitive.*

Proof. Let D_n be a dense subset of X_n for $n = 1, 2, \ldots$. For every integer n let us denote by z_n some fixed point of X_n and then consider a countable subset

$$\mathbb{D}_n = D_1 \times D_2 \times \cdots \times D_n \times \{z_{n+1}\} \times \cdots$$

of $\mathbb{X}$. Let $\mathbb{D} = \bigcup_{n=1}^{\infty} \mathbb{D}_n$. We shall show that $\mathbb{D}$ is dense in $\mathbb{X}$, i.e. $\mathbb{D} \cap V \neq \emptyset$ for an arbitrary non - empty open set $V \subset \mathbb{X}$. Of course, we can assume that $V = U_1 \times \cdots \times U_n \times X_{n+1} \times \cdots$ where U_i is an open subset of X_i for $i = 1, \cdots, n$. Let $d_i \in A_i \cap U_i$ for every $i = 1, \cdots, n$. Clearly for $\mathbf{d} = (d_1, \cdots, d_n, z_{n+1}, \cdots)$ we have $\mathbf{d} \in \mathbb{D} \cap V$.

In the sequel we are going to construct an element $t \in \mathbb{X}$ whose forward orbit $O_+(\mathbf{g})(t)$ is a dense set in $\mathbb{X}$ (*the hyper cyclic element of* $\mathbf{g}$). Let us define for this purpose for $\mathbf{d} \in \mathbb{D}$ the *weight* $w(\mathbf{d})$ of $\mathbf{d}$ by means of the formula

$$w(\mathbf{d}) = \max\{i | \mathbf{d}(i) \neq z_i\}.$$

It is easy to verify that the set $\mathbb{D}$ is countable and, moreover, that it can be written as a sequence $\mathbb{D} = \{\mathbf{d}_1, \mathbf{d}_2, \cdots\}$ such that the condition $w(\mathbf{d}_n) \leq n$ holds for every $n \in \mathbb{N}$. So we can write $\mathbf{d}_n = (d_{n1}, \ldots, d_{nn}, z_{n+1}, \ldots)$.

Further, let

$$M = \bigcup_{k=1}^{\infty} (\mathbb{I}(k) \times \{p_k\})$$

and let ν^∞ be the function $\nu^\infty : M \to \mathbb{N}$ defined by $\nu^\infty(i, p_k) = \nu^{p_k}(i)$. Clearly, ν^∞ is injective and $\nu^\infty(\mathbb{I}(k) \times \{p_k\}) \cap \nu^\infty(\mathbb{I}(l) \times \{p_l\}) = \emptyset$ for $k \neq l$, since $\nu^\infty(\mathbb{I}(k)) = \nu^{p_k}(\mathbb{I}(k))$ for every $k \in \mathbb{N}$.

Next we can give the construction of the element $t \in \mathbb{X}$:

(a) $t(j) = z_j \in X_j$ if $j \notin \nu^\infty(M)$.

(b) if $j \in \nu^\infty(M)$, then $j = \nu^{p_k}(i)$ for some k and $i \in \mathbb{I}(k)$. Choose now $t(j) \in h_j^{-1}(d_{ki})$, where $h_j = g_i \circ g_{\nu(i)} \circ \cdots \circ g_{\nu^{p_k - 1}(i)}$.

It remains to show that the forward orbit $O_+(\mathbf{g})(t) = \{\mathbf{g}^n(t) | n \in \mathbb{N}\}$ is a dense set. Let, as above, V be an open subset of $\mathbb{X}$ and $V = U_1 \times \cdots \times U_n \times X_{n+1} \times \cdots$. Then, for some $k \geq n$ we should have $\mathbf{d}_k = (d_{k1}, \dots, d_{kk}, z_{k+1}, \dots) \in \mathbb{D} \cap V$, where $d_{ki} \in U_i$. It follows now by the construction of the element t that $\mathbf{g}^{p_k}(t)$ has the form

$$\mathbf{g}^{p_k}(t) = (d_{k1}, d_{k2}, \dots, d_{kk}, \omega_{k+1}, \dots),$$

where ω_n for $n \geq k+1$ is some (arbitrary) element of X_n. Clearly, $\mathbf{g}^{p_k}(t) \in V$.

3. Here we apply the above considerations to the space $\mathbb{R}^{\mathbb{R}}$.

We may use, for example, the well known fact that the spaces $\mathbb{R}^{\mathbb{R}}$ and $\left(\mathbb{R}^{\mathbb{R}}\right)^{\aleph_0}$ are homeomorphic [3]. Let us denote by f the homeomorphism $f : \mathbb{X} \to \mathbb{R}^{\mathbb{R}}$, where $\mathbb{X} = \prod\limits_{n=1}^{\infty} X_n$ with $X_n = \mathbb{R}$ for every positive integer n. Note that we have a wide choice of the onto maps $g_n : X_{\nu(n)} \to X_n$ from Section 1 – for example, we may put $g_n = id$. It follows by Theorem 2.4 that there is a transitive shift operator $\mathbf{g} : \mathbb{X} \to \mathbb{X}$. Then the map $\mathbf{h} : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ defined by $\mathbf{h}(x) = f^{-1} \circ \mathbf{g} \circ f(x)$ is transitive.

Speaking in general, Section 2 gives different ways to obtain transitive maps on $\mathbb{R}^{\mathbb{R}}$. We offer here one more example of a transitive map on $\mathbb{R}^{\mathbb{R}}$.

In what follows we denote by A_n the set $\{n\} \times (-n, n)$ and let $\mathbb{A} = \bigcup\limits_{n=1}^{\infty} A_n$. Obviously, the set $\mathbb{A}$ has the same cardinality as $\mathbb{R}$ and, hence, the spaces $\mathbb{R}^{\mathbb{A}}$ and $\mathbb{R}^{\mathbb{R}}$ are homeomorphic. Let $u : \mathbb{R}^{\mathbb{A}} \to \mathbb{R}^{\mathbb{R}}$ be a homeomorphism. Moreover, note that for every integer n, $\mathbb{R}^{\mathbb{R}} \overset{top}{=} \mathbb{R}^{A_n}$. Since $\mathbb{A}$ is a disjoint sum of A_n's, we should have $\mathbb{R}^{\mathbb{A}} = \prod\limits_{n=1}^{\infty} \mathbb{R}^{A_n}$ [3]. Therefore, if we put $X_n = \mathbb{R}^{A_n}$ then $\mathbb{R}^{\mathbb{A}} = \prod\limits_{n=1}^{\infty} X_n$ so one can apply Theorem 2.4 to obtain a shift operator $\mathbf{v} : \mathbb{R}^{\mathbb{A}} \to \mathbb{R}^{\mathbb{A}}$. Here one can define the maps g_n as follows: for $x = (x_\alpha) \in \mathbb{R}^{(-\nu(n), \nu(n))} = X_{\nu(n)}$, we let $g_n(x) = \left(x_{\frac{n\alpha}{\nu(n)}}\right)$.

Now, one can put $\mathbf{w} : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ by letting $\mathbf{w} = u^{-1} \circ \mathbf{v} \circ u$ to obtain a transitive map on $\mathbb{R}^{\mathbb{R}}$.

Note that there is a lot of publications about transitive functions (see the excellent survey [4]).

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ТРАНЗИТИВНИ ОПЕРАТОРИ В $\mathbb{R}^{\mathbb{R}}$

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Нека X е топологично пространство и $Y \subset X$. Ще казваме, че операторът (функцията) $f : Y \to X$ е *тразитивна*, ако съществува елемент $t \in Y$, положителната полуорбита

$$O_{+}f(t) = \{f^{n}(t) | n \in N\}$$

на който е навсякъде гъсто подмножество на X. В тази бележка разглеждаме пример на транзитивни оператори, дефинирани в неизброимото произведение $X = \mathbb{R}^{\mathbb{R}}$ (да напомним, че X има неизброимо тегло).