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HIGH AND LOW INTENSITY OF REPETITIONS IN A FINITE M/G/1/0 QUEUEING SYSTEM WITH REPEATED CALLS*

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A Finite M/G/1/0 queueing system with repeated calls is studied in the cases when the intensity of repetitions converges to zero and to infinity, respectively. The obtained asymptotical formulas are compared with those in the corresponding classical systems without repeated calls. Numerical data illustrating the obtained results are presented. It is shown that the assumption of a finite number of customers in a single-line queueing system with repeated calls influences considerably its asymptotical properties when the intensity of repetitions is small or high.

1. Introduction. There is quite a number of different single-server repeated orders queueing systems. The main peculiarity of these systems is the assumption that a customer arriving when the server is busy, repeats his demand after some delay. This holds true especially in a teletraffic theory, as it is well known that a telephone subscriber who obtains an engaged signal usually repeats the call until the required connection is made.

As in a real situation the number of subscribers is finite, the investigation of a finite system with repeated calls is of special interest to practice. This assumption as well as the presence of repeated calls complicates the investigation of the system and the expressions for its characteristics. Formulas for some of these characteristics are obtained in previous works of the author ([1], [2]). The purpose of the present paper is to study asymptotical properties of the obtained formulas and to compare them with the corresponding more simple formulas in classical systems (with losses or with queue).

2. Model description and previous results. We consider a finite single-line queueing system with N customers. These customers are identified as sources of primary orders (calls, demands,...). Each such source produces a Poisson process of primary calls with intensity λ. If the server is free at the instant of a primary call arrival it begins service immediately. Otherwise, if the channel is busy, then it forms a source of repeated calls. Such a source produces a Poisson process of repeated calls with intensity μ. If an incoming repeated call finds a free line, then it begins service and, as well as a primary call, after service completion becomes again a source of primary calls. Otherwise, if the line is engaged at the moment of repeated call arrival, then the system state does not change.

The service time distribution function is $F(x)$ both for primary and repeated calls. The intervals between repeated trials, as well as between primary ones and the service times, are assumed to be mutually independent.

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Let

$$S(x) = F'(x),\ \overline{S}(s) = \int_0^\infty e^{-sx} dF(x) = \int_0^\infty e^{-sx}\ S(x)dx\ \ \nu^{-1} = -\overline{S}'(0),$$

$C(t)$ be the number of busy lines at time t, and $N(t)$ - the number of repeated calls sources at time t (a sort of queue). The process $(N(t), C(t))$ in steady state was studied in previous works of the author ([1, 2]). Because of the heavy expressions and the limited paper size, we not recapitulate the formulas for the joint distribution of the channel state $C(t)$ and the queue length $N(t)$ in steady state

$$p_{ij} = \lim_{t\to\infty} P\left\{C(t) = i,\ N(t) = j\right\},\ \ i = 0, 1,\ \ \ j = 0, 1, \ldots, N-1.$$

For the same reasons we omit the formulas for the variance DN of the queue length and, thus, the obtained asymptotical properties and numerical data.

The following theorem holds true [1].

Theorem 1. *The stationary distribution of the channel state*

$$P_i = \lim_{t\to\infty} P\left\{C(t) = i, \right\}, i = 0, 1,$$

and the mean EN of the queue length in steady state

$$EN = \lim_{t\to\infty}\left(\sum_{n=0}^{N-1} np_{0n} + \sum_{n=0}^{N-1} np_{1n}\right),$$

have the form

$$P_1 = \nu^{-1}\psi_{N-1}C, \quad P_0 = 1 - P_1, \tag{1}$$

$$EN = \left[(N-1)\nu^{-1}\psi_{N-1} + \frac{(1 - S_1)(\mu - \lambda)\psi_{N-2}}{\mu\lambda}\right] C, \tag{2}$$

where

$$C = N\lambda\mu\left[\mu\left(1 + N\lambda\nu^{-1}\right)\psi_{N-1} + (1 - S_1)(\mu - \lambda)\psi_{N-2}\right]^{-1}, \tag{3}$$

$$\psi_n = (-1)^n \binom{N-1}{n}\frac{S_{N-1}}{S_{N-n-1}}\left(1 + A_n + B_n + C_n\right),\ \ n = 0, 1, \ldots, N-1 \tag{4}$$

and A_n, B_n and C_n are given by the recurrent relations

$$A_0 = B_0 = C_0 = 0, \tag{5}$$

$$A_n = \frac{1 - S_{N-n}}{S_{N-n}}\frac{a_n}{(N-n)\mu}\left(A_{n-1} + B_{n-1}\right), \tag{6}$$

$$B_n = \frac{a_0}{a_{n-1}}\left(A_{n-1} + C_{n-1}\right) + B_{n-1}, \tag{7}$$

$$C_n = \frac{1 - S_{N-n}}{S_{N-n}}\frac{a_n}{(N-n)\mu}\left(1 + C_{n-1}\right),\ \ n = 1, 2, \ldots, N-1, \tag{8}$$

$$a_n = (N-n)\lambda + n\mu, S_n = \overline{S}(n\lambda),\ \ n = 0, 1, \ldots, N-1.$$

3. Limit properties when the repeated orders intensity $\mu \to \infty$. In a real situation subscribers repeat calls practically immediately. So an investigation of asymptotic behavior of the repeated orders queueing system characteristics under high intensity of repetition is of special interest to practice.

Theorem 2. *When $\mu \to \infty$, the asymptotical values $P_i(\infty)$ and $EN(\infty)$ of the channel state distribution P_i, $i = 1, 2$ and the mean EN of the queue length in steady state are*

$$P_1(\infty) = \lim_{\mu\to\infty} P_1 = 1 - \hat{e}, \quad P_0(\infty) = 1 - P_1(\infty) = \hat{e}, \tag{9}$$

$$EN(\infty) = \lim_{\mu\to\infty} EN = N - (1 - \hat{e})\left(1 + \frac{1}{\nu^{-1}\lambda}\right), \tag{10}$$

where

$$\hat{e} = \left(1 + N\lambda\nu^{-1}\sum_{j=1}^{N-1}\binom{N-1}{j}d_j\right)^{-1}, \tag{11}$$

$$d_j = \begin{cases} \dfrac{1-S_1}{S_1}\dfrac{1-S_2}{S_2}\cdots\dfrac{1-S_j}{S_j}, & \text{for } j = 1, 2, \ldots, N-1, \\ 1, \quad \text{for } j = 0. \end{cases} \tag{12}$$

Proof. From formulas (5) – (8), by means of mathematical induction, it is not too difficult to prove that

$$\lim_{\mu\to\infty} A_n = \lim_{\mu\to\infty} B_n = \lim_{\mu\to\infty} C_0 = 0, \; n = 0, 1, \ldots, N-1,$$

$$\lim_{\mu\to\infty} C_n = \sum_{l=1}^{n}\frac{n(n-1)\ldots l}{(N-n)\ldots(N-l)}\frac{1-S_{N-n}}{S_{N-n}}\cdots\frac{1-S_{N-l}}{S_{N-l}}, \; n = 1, 2, \ldots, N-1.$$

So, taking limit in (4), we get

$$\psi_0 = 1, \quad \lim_{\mu\to\infty}\psi_n = (-1)^n\binom{N-1}{n}\frac{S_{N-1}}{S_{N-n-1}}\left(1 + \sum_{l=1}^{n}\frac{n(n-1)\ldots l}{(N-n)\ldots(N-l)}\frac{d_{N-l}}{d_{N-n-1}}\right),$$

$n = 1, 2, \ldots, N-1$.

Having from here expressions for $\lim\limits_{\mu\to\infty}\psi_{N-1}$ and $\lim\limits_{\mu\to\infty}\psi_{N-2}$ and taking the limit as $\mu \to \infty$ in (3) and (1) – (2) consecutively, we obtain first that

$$\lim_{\mu\to\infty} C = \frac{(-1)^{N-1}N\lambda\hat{e}}{S_{N-1}},$$

where $\hat{e}$ is given by (11) – (12), and from here formulas (9) – (10). □

When $\mu \to \infty$, it is natural to compare the obtained asymptotical formulas with the corresponding ones in a finite $M/G/1/\infty$ system, i.e. a classical M/G/1/∞ system with service time distribution function $F(x)$ and N customers, each one producing a Poisson process of orders with intensity λ. This system was studied in [3]. The comparison shows that when $\mu \to \infty$, the stationary distribution of the channel state P_i, $i = 1, 2$, in our system with repeated calls converges to the stationary distribution of the channel state $\widetilde{P}_i$, $i = 1, 2$, in the system without repetitions, but this is not true for the stationary characteristics of the queue length. In particular, if $E\widetilde{N}$ is the mean of the stationary queue length in the system without repetition, then

$$EN(\infty) = \lim_{\mu\to\infty} EN = E\widetilde{N} - (1 - \hat{e}).$$

Let us note that if the considered systems were not finite, then when $\mu \to \infty$ not only the stationary distribution of the channel state, but also the distribution of the queue length in the system with repeated calls converges to the corresponding distribution in the system without repeated calls [4].

4. Limit properties when the repeated orders intensity $\mu \to 0$. Intuitive reasons prompt, that in the case $\mu \to 0$ the repeated orders queueing system may be considered as the corresponding $M/G/1/0$ system with losses. As this system is classical in teletraffic theory and the formulas therein are much simpler, the question of comparison between both systems is of interest.

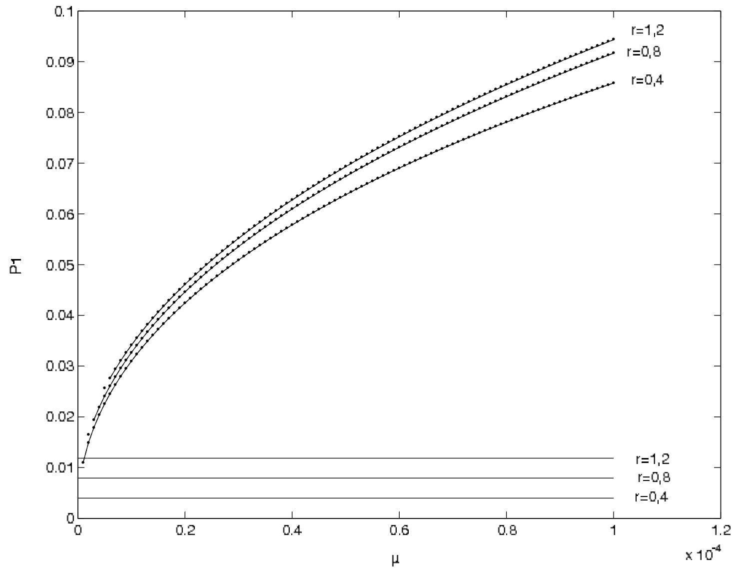

Fig. 1

Theorem 3. *When $\mu \to 0$, the asymptotical values $P_i(0)$ and $EN(0)$ of the channel state distribution P_i, $i = 1, 2$, and the mean EN of the queue length in steady state are given by*

$$P_1(0) = \lim_{\mu \to 0} P_1 = \frac{\lambda\nu^{-1}}{1 + \lambda\nu^{-1}}, \quad P_0(0) = 1 - \lim_{\mu \to 0} P_1 = \frac{1}{1 + \lambda\nu^{-1}} \tag{13}$$

$$EN(0) = \lim_{\mu \to 0} EN = N - 1. \tag{14}$$

Proof. As in the proof of the Theorem 2, from formulas (5) – (8) and by means of mathematical induction we prove, that

$$\lim_{\mu \to 0} C_0 = \lim_{\mu \to 0} B_0 = \lim_{\mu \to 0} B_1 = \lim_{\mu \to 0} A_0 = \lim_{\mu \to 0} A_1 = \lim_{\mu \to 0} A_2 = 0,$$

$$\lim_{\mu \to 0} \mu^n C_n = \lambda^n \frac{1 - S_{N-n}}{S_{N-n}} \frac{1 - S_{N-(n-1)}}{S_{N-(n-1)}} \cdots \frac{1 - S_{N-1}}{S_{N-1}}, \quad n = 1, 2, \ldots, N-1,$$

$$\lim_{\mu\to 0}\mu^{n-1}B_n = \frac{N}{N-(n-1)}\lim_{\mu\to 0}\mu^{n-1}C_{n-1} =$$

$$= \lambda^{n-1}\frac{N}{N-(n-1)}\frac{1-S_{N-(n-1)}}{S_{N-(n-1)}}\frac{1-S_{N-(n-2)}}{S_{N-(n-2)}}\cdots\frac{1-S_{N-1}}{S_{N-1}},\ n=2,3,\ldots,N-1,$$

$$\lim_{\mu\to 0}\mu^{n-1}A_n = \lambda^{n-1}\sum_{l=1}^{n-2}\frac{N}{N-l}\frac{1-S_{N-1}}{S_{N-1}}\cdots\frac{1-S_{N-l}}{S_{N-l}}\frac{1-S_{N-(l+2)}}{S_{N-(l+2)}}\cdots\frac{1-S_{N-n}}{S_{N-n}},$$
$$n=3,4,\ldots,N-1.$$

Taking limit as $\mu\to 0$ in (4) and taking into account the above formulas, we get

$$\lim_{\mu\to 0}\mu^n\psi_n = (-1)^n\binom{N-1}{n}\frac{S_{N-1}}{S_{N-n-1}}\lambda^n\frac{1-S_{N-n}}{S_{N-n}}\frac{1-S_{N-(n-1)}}{S_{N-(n-1)}}\cdots\frac{1-S_{N-1}}{S_{N-1}},$$
$$n=1,2,\ldots,N-1,\quad \psi_0=1.$$

Using this formula for $n=N-1,\ N-2$ and after taking limit in (3), we have

$$\lim_{\mu\to 0}\frac{C}{\mu^{N-1}} = \left[(1+\lambda\nu^{-1})\lambda^{N-2}(-1)^{N-1}\frac{1-S_1}{S_1}\cdots\frac{1-S_{N-2}}{S_{N-2}}(1-S_{N-1})\right]^{-1}.$$

From here and formulas (1) – (2) we obtain the results (13) – (14) of Theorem 3. $\square$

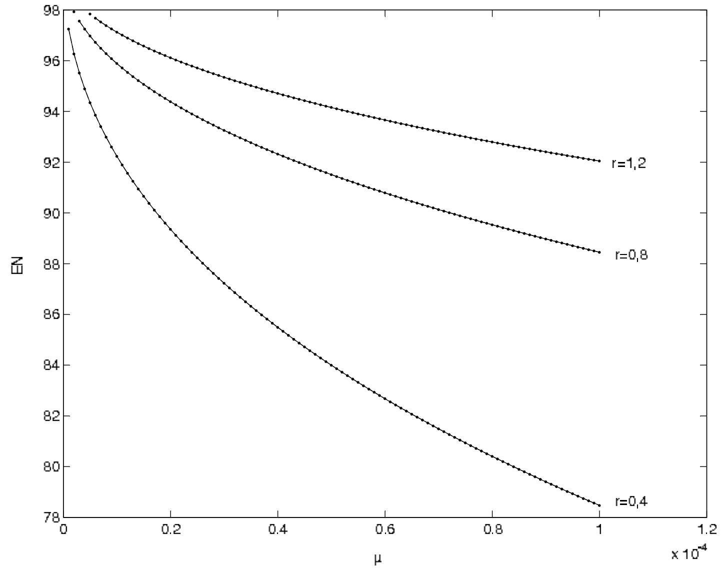

Fig. 2

Let us notice that when $\mu\to 0$, the limit distribution (13) of the channel state does not depend on the number N of the customers. It is just the stationary distribution of the channel state in the corresponding classical system with losses, i.e. a $M/G/1/0$ system with input flow rate λ, service time distribution $F(x)$ and losses [5].

5. Numerical data. Numerical analysis of the system is performed using the software system MATLAB [6]. Fig. 1 and Fig. 2 show correspondingly the dependence of the probability P_1 (1) and the mean EN (2) on μ, when μ is near 0. The values of P_1 and EN are calculated in three cases of service times distribution $F(x)$: a) exponential distribution with parameter 1 (point lines), b) Erlang Distribution with parameters 4,4 (dashed lines) and c) constant 1 (solid lines). As it is shown by the drawings, these three lines coincide entirely, wich means that for small values of μ the service time distribution does not affect the results. The straight lines in Fig. 1 correspond to the boundary values of P_1 (13), as depending on r. We assume that $\nu = 1$, $N = 100$ and consider three values of the parameter $r = N\lambda\nu^{-1}$: 0,4, 0,8 and 1,2.

The numerical data confirm the theoretical limits as $\mu \to 0$ obtained in this article. The parameter r influences the convergence rate of EN and more weakly that one of P_1.

We do not apply here the corresponding graphics for big values of μ. Let us note that the numerical analysis shows that for big μ the convergence to the obtained limits is much faster than in the case of small μ, and the dependence on the type of the service times distribution is essential.

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ВИСОКА И НИСКА ИНТЕНЗИВНОСТ НА ПОВТОРЕНИЯТА В M/G/1/0 СИСТЕМА НА МАСОВО ОБСЛУЖВАНЕ С КРАЕН ИЗТОЧНИК И ПОВТОРНИ ЗАЯВКИ

Велика Илиева Драгиева

Разгледана е M/G/1/0 система на масово обслужване с краен източник и повторни заявки в случаите, когато интензивността на повторенията клони съответно към нула или към безкрайност. Получените асимптотични формули са сравнени с тези в съответстващите класически системи без повторни заявки. Представени са числени данни, илюстриращи получените резултати. Установено е, че крайният брой клиенти в разглежданата еднолинейна система на масово обслужване с повторни заявки влияе значително върху нейните асимтотични свойства при висока и при ниска интензивност на повторенията.