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SET-VALUED OPTIMIZATION PROBLEM INVOLVING SINGLE-VALUED EQUALITY CONSTRAINTS*

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The set-valued optimization problem $\min_{C} F(x)$, $G(x) \cap (-K) \neq \emptyset$, h(x) = 0 is considered, where $C \subset \mathbb{R}^{m}$ and $K \subset \mathbb{R}^{p}$ are closed convex cones, $F : \mathbb{R}^{n} \to \mathbb{R}^{m}$ and $G : \mathbb{R}^{n} \to \mathbb{R}^{p}$ are set-valued functions, and $h : \mathbb{R}^{n} \to \mathbb{R}^{q}$ is C^{1} -function. Two type of solutions, namely w-minimizers (weakly efficient point) and *i*-minimizers (isolated minimizers), are treated. In terms of the Dini set-valued directional derivative firstorder necessary conditions a point to be a w-minimizers, and first-order sufficient conditions a point to be an *i*-minimizer are established, both in primal and dual form.

1. Introduction. The constrained set-valued optimization problem (*svp*)

 $\min_C F(x), \quad G(x) \cap (-K) \neq \emptyset, \quad h(x) = 0,$

is considered, where $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^p$ are set-valued functions (svf) with non-empty values, $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones, and $h : \mathbb{R}^n \to \mathbb{R}^q$ is a C^1 -function. First order optimality conditions, both in primal and dual form, in terms of the Dini set-valued directional derivative are derived. Recently, optimality conditions for svp are studied mainly by mean of epiderivatives, e. g. in [5], [6] and [2]. We consider the optimality conditions based on directional derivatives as certain alternative of those based on epiderivatives.

2. Preliminaries.

(1)

The dual pairing in \mathbb{R}^k is denoted $\langle \cdot, \cdot \rangle$. The notations B_k and \overline{B}_k are used for the open and closed unit balls in \mathbb{R}^k , and $B_k(x^0)$ and $\overline{B}_k(x^0)$ for the open and closed unit balls with center x^0 .

For a given closed convex cone $M \subset \mathbb{R}^k$ its positive polar cone is defined by $M' = \{\xi \in \mathbb{R}^k \mid \langle \xi, x \rangle \geq 0 \text{ for all } y \in M \}$. When $x^0 \in M$, we put $M'[x^0] = \{\xi \in M' \mid \langle \xi, x^0 \rangle = 0\}$ and $M[x^0] = (M'[x^0]))'$. It holds $M \subset M[x^0]$.

When \mathbb{R}^k is considered with a concrete norm, then the distance from a point $x \in \mathbb{R}^k$ to a set $A \subset \mathbb{R}^k$ is given by $d(x, A) = \inf\{||x - y|| \mid a \in A\}$. The oriented distance from x to A is defined by $D(x, A) = d(x, A) - d(x, \mathbb{R}^k \setminus A)$. We define the oriented distance D(P, A)from a set $P \subset \mathbb{R}^k$ to the set $A \subset \mathbb{R}^k$ by putting $D(P, A) = \inf\{D(x, A) \mid x \in P\}$.

Let $M \subset \mathbb{R}^k$ be a cone and let a be a real number. Then, we put $M(a) = \{x \in \mathbb{R}^k \mid D(x, M) \leq a ||x||\}$. The weakly efficient frontier (w-frontier) w-Min_MA and the

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properly efficient frontier (*p*-frontier) p-Min_MA of A are defined by w-Min_MA = { $x \in A \mid A \cap (x - \text{int } M) = \emptyset$ } and p-Min_MA = { $x \in A \mid \exists a \in (0, 1) : A \cap (x - M(a)) = \{x\}$ } respectively.

The set of the feasible points of svp(1) is $\mathcal{G} = \{x \in \mathbb{R}^n \mid G(x) \cap (-K) \neq \emptyset, h(x) = 0\}$. Further, $\mathcal{N}(x^0)$ denotes the family of the neighbourhoods of x^0 . We deal with local solutions of (1), which in any case are pairs $(x^0, y^0), y^0 \in F(x^0)$, with x^0 feasible. We use the following concepts of solutions for problem (1). The pair $(x^0, y^0), x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, is said a *w*-minimizer (weakly efficient point) if there exists $U \in \mathcal{N}(x^0)$ such that $x \in U \cap \mathcal{G}$ implies $F(x) \cap (y^0 - \operatorname{int} C) = \emptyset$ (then necessary $y^0 \in w$ - $Min_CF(x^0)$). The pair (x^0, y^0) is said an *i*-minimizer (isolated minimizer) if there exists $U \in \mathcal{N}(x^0)$ and a constant A > 0 such that $D(F(x) - y^0, -C) \ge A ||x - x^0||$ and $y^0 \in p$ - $\operatorname{Min}_C F(x^0)$ for $x \in U \cap \mathcal{G}$ (the concept of *i*-minimizer is norm-independent, since all norms in finite-dimensional spaces are equivalent).

The $svf \ \Phi : \mathbb{R}^n \rightsquigarrow \mathbb{R}^k$ is said locally Lipschitz at $x^0 \in \mathbb{R}^n$, if there exists $U \in \mathcal{N}(x^0)$ and a constant L > 0, such that for $x^1, x^2 \in U$ it holds $\Phi(x^2) \subset \Phi(x^1) + L ||x^2 - x^1|| \overline{B}_k$. The svf Φ is said locally Lipschitz, if it is locally Lipschitz at each $x^0 \in \mathbb{R}^n$. Given a cone $M \subset \mathbb{R}^k$, we say that Φ is locally M-Lipschitz at x^0 if the $svf \ x \rightsquigarrow \Phi(x) + M$ is locally Lipschitz at x^0 . The svf Φ is said locally M-Lipschitz, if it is locally M-Lipschitz at each point x^0 .

The convex cone $M \subset \mathbb{R}^k$ is said pointed, if $(-M) \cap M = \{0\}$.

Our aim is to obtain optimality conditions for svp (1) in terms of Dini derivatives. For the svf $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ the Dini derivative of Φ at $(x^0, y^0), y^0 \in \Phi(x^0)$, in direction $u \in \mathbb{R}^n$ is defined

$$\Phi'(x^0, y^0; u) = \underset{t \to 0^+}{\text{Limsup}} \frac{1}{t} \left(\Phi(x^0 + tu) - y^0 \right).$$

3. Problems without equality constraints. Without the equality constraints the problem reduces to

(2)
$$\min_C F(x), \quad G(x) \cap (-K) \neq \emptyset.$$

This problem is investigated in [3]. There, generalizing [4] from vector to set-valued problem, and [1] from unconstrained to constrained problem, the following results are established.

Theorem 1. [Necessary Conditions] Consider svp (2) with $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ closed convex cones, $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^p$ svf. Let the pair (x^0, y^0) , $x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, be a w-minimizer of svp (2) and let $z^0 \in G(x^0) \cap (-K)$. Then,

$$\forall u \in \mathbb{R}^m : (F \times G)'(x^0, (y^0, z^0); u) \cap (-(\operatorname{int} C \times \operatorname{int} K[-z^0]) = \emptyset.$$

Theorem 2. [Sufficient Conditions] Consider sup (2) with $C \subset \mathbb{R}^m$ pointed closed convex cone, $K \subset \mathbb{R}^p$ closed convex cone, $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ locally C-Lipschitz suf, and $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ locally Lipschitz suf. Suppose that the pair $(x^0, y^0), x^0 \in \mathbb{R}^n, y^0 \in F(x^0),$ is such that $y^0 \in p$ -Min_C $F(x^0)$, and there exists $z^0 \in G(x^0) \cap (-K)$ for which

$$\forall u \in \mathbb{R}^n \setminus \{0\} : (F \times G)'(x^0, (y^0, z^0); u) \cap (-(C \times K[-z^0])) = \emptyset.$$

Suppose also that the svf G satisfies the following condition:

Then, (x^0, y^0) is an i-minimizer of svp (2).

Let us point out that in [3] it was given an example showing that without condition $\mathbb{G}(x^0, x^0)$ Theorem 2 is not true. Though condition $\mathbb{G}(x^0, z^0)$ does not appear in Theorem 3, the interesting applications of this theorem could be those when $G(x) \cap (-K)$ possesses points near z^0 . Indeed, suppose that $\forall \ell > 0 : \exists U \in \mathcal{N}(x^0) : \forall x \in U \cap \mathcal{G} : G(x) \cap \ell || x - x^0 || \bar{B}_p(x^0) \cap (-K) = \emptyset$. Then, $(F \times G)'(x^0, (y^0, z^0); u) = \emptyset$ for all $u \in \mathbb{R}^n$, and condition (3) is satisfied for arbitrary svf F.

4. Problems with equality constraints. In this section we generalize the results from the previous one from the restricted problem (2) to the more generalize problem (1). In fact, we prove the following results.

Theorem 3. [Necessary Conditions] Consider sup (1) with $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ closed convex cones, $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^p$ locally Lipschitz suf, and h : $\mathbb{R}^n \to \mathbb{R}^q$ a C^1 -function such that at the point $x^0 \in \mathbb{R}^n$ the vectors $h'_1(x^0), \ldots, h'_q(x^0)$, where h_1, \ldots, h_q are the components of h, are linearly independent. Let the pair (x^0, y^0) , $y^0 \in F(x^0)$, be a w-minimizer of sup (1) and let $z^0 \in G(x^0) \cap (-K)$. Then,

(3) $\forall u \in \ker h'(x^0) : (F \times G)'(x^0, (y^0, z^0); u) \cap (-(\operatorname{int} C \times \operatorname{int} K[-z^0]) = \emptyset.$

Theorem 4. [Sufficient Conditions] Consider svp (1) with $C \subset \mathbb{R}^m$ pointed closed convex cone, $K \subset \mathbb{R}^p$ closed convex cone, $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^p$ locally Lipschitz svf, and $h : \mathbb{R}^n \to \mathbb{R}^q$ a C^1 -function such that at the feasible point $x^0 \in \mathbb{R}^n$ the vectors $h'_1(x^0), \ldots, h'_q(x^0)$ are linearly independent. Suppose that the pair (x^0, y^0) , $y^0 \in F(x^0)$, is such that $y^0 \in p$ -Min_C $F(x^0)$, and there exists $z^0 \in G(x^0) \cap (-K)$ for which

(4)
$$\forall u \in \ker h'(x^0) \setminus \{0\} : (F \times G)'(x^0, (y^0, z^0); u) \cap (-(C \times K[-z^0])) = \emptyset.$$

Suppose also that the svf G satisfies the following condition: $\Box U = \Delta G = \Delta G$

$$\mathbb{G}(x^0, z^0): \qquad \qquad \exists U \in \mathcal{N}(x^0) : \exists \ell > 0 : \forall x \in U : \\ G(x) \cap (-K) \neq \emptyset \Rightarrow G(x) \cap \ell ||x - x^0|| \bar{B}_p(z^0) \cap (-K) \neq \emptyset$$

Then, (x^0, y^0) is an i-minimizer of svp (1).

Remark. Condition (4), which can be regarded as a primal form condition, can be substituted by the equivalent dual form condition

(5)
$$\forall u \in \ker h'(x^0) \setminus \{0\} : \forall, (\bar{y}^0, \bar{z}^0) \in (F \times G)'(x^0, (y^0, z^0); u) : \\ \exists (\xi, \eta) \in C' \times K'[-z^0], (\xi, \eta) \neq (0, 0) : \langle \xi, \bar{y}^0 \rangle + \langle \eta, \bar{z}^0 \rangle > 0 .$$

A similar dual form admits condition (3), which differs from (5) only by the last expression, which should be replaced by the non-strict inequality $\langle \xi, \bar{y}^0 \rangle + \langle \eta, \bar{z}^0 \rangle \geq 0$.

Now, we prove Theorem 4 by transforming problem (1) to a problem without equality constraints. The proof of Theorem 3 can be obtained similarly.

Proof of Theorem 4. Let the vector $\bar{u}^j \in \mathbb{R}^n$, j = 1, ..., q, be determined by the system of equations

(6)
$$h'_k(x^0)\bar{u}^j = 0 \text{ for } k \neq j, \text{ and } h'_j(x^0)\bar{u}^j = 1.$$

For each j = 1, ..., q, equalities (6) constitute a system of linear equations with respect to the components of \bar{u}^j , which due to the linear independence of $h'_1(x^0), ..., h'_q(x^0)$ has a unique solution. Moreover, the vectors $\bar{u}^1, ..., \bar{u}^q$ solving this system are linearly independent and \mathbb{R}^n is decomposed into a direct sum $\mathbb{R}^n = L \oplus L'$, where $L = \ker h'(x^0)$ 192 and $L' = \lim \{\bar{u}^1, \ldots, \bar{u}^q\}$. Let u^1, \ldots, u^{n-q} be n-q linearly independent vectors in $L = \ker h'(x^0)$. We consider the system of equations

$$h_k(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j) = 0, \quad k = 1, \dots, q$$

Taking $\tau_1, \ldots, \tau_{n-q}$ as independent variables and $\sigma_1, \ldots, \sigma_q$ as dependent variables, we see that this system satisfies the requirements of the implicit function theorem at the point $\tau_1 = \cdots = \tau_{n-q} = 0$, $\sigma_1 = \cdots = \sigma_q = 0$ (at this point $h_k(x^0) = 0$ because x^0 is feasible, and the Jacobian $\partial h/\partial \sigma$ is the unit matrix and, hence, it is non degenerate). The implicit function theorem gives that in a neighbourhood of x^0 given by $|\tau_i| < \bar{\tau}$, $i = 1, \ldots, n-q$, $|\sigma_j| < \bar{\sigma}$, $j = 1, \cdots, q$, this system possesses a unique solution $\sigma_j = \sigma_j(\tau_1, \ldots, \tau_{n-q})$, $j = 1, \ldots, q$. The functions $\sigma_j = \sigma_j(\tau_1, \ldots, \tau_{n-q})$ are C^1 , and

$$\sigma_j|_{\tau^0} = \sigma_j(0, \dots, 0) = 0, \quad j = 1 \dots, q,$$
$$\frac{\partial \sigma_j}{\partial \tau_i}\Big|_{\tau^0} = 0, \quad j = 1, \dots, q, \quad i = 1, \dots, n - q,$$

where $\tau^0 = (0, \ldots, 0)$. It is clear that (x^0, y^0) is a *w*-minimizer or *i*-minimizer of problem (1) if and only if (τ^0, y^0) is, respectively, a *w*-minimizer or *i*-minimizer of the problem (7) $\min_C \bar{F}(\tau_1, \ldots, \tau_{n-q}), \quad \bar{G}(\tau_1, \ldots, \tau_{n-q}) \cap (-K) \neq \emptyset,$

where

$$\bar{F}(\tau_1, \dots, \tau_{n-q}) = F(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^q \sigma_j(\tau_1, \dots, \tau_{n-q}) \,\bar{u}^j),$$

$$\bar{G}(\tau_1, \dots, \tau_{n-q}) = G(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^q \sigma_j(\tau_1, \dots, \tau_{n-q}) \,\bar{u}^j).$$

Applying Theorem 2 to problem (7) and arguing routinely that

$$(\bar{F}\times\bar{G})'(\tau^0,(y^0,z^0);\tau)=(F\times G)'(x^0,(y^0,z^0);u)\,,$$

where $u = \sum_{i=1}^{n-q} \tau_i u^i$, we get the assertion of Theorem 4. Still, let us point out that the proof of the above equality uses the Lipschitz property of F and G.

A particular case of (1) is the single-valued vector optimization problem

(8) $\min_C f(x), \quad g(x) \in -K, \quad h(x) = 0.$

Then, from Theorems 3 and 4 we get the following theorem, which generalizes the results of [4] from problems with only inequality constraints to problems with both inequality and equality constraints.

Theorem 5. Consider problem (8) with f and g locally Lipschitz functions, h a C^1 -function, C pointed closed convex cone, and K closed convex cone. Let x^0 be a feasible point and suppose that the vectors $h'_1(x^0), \ldots, h'_q(x^0)$, being the components of $h'(x^0)$, are linearly independent.

(Necessary Conditions) Let x^0 be a w-minimizer of problem (8). Then, for each $u \in \ker h'(x^0) \setminus \{0\}$ the following condition is satisfied:

$$\forall (y^0, z^0) \in (f, g)'(x^0; u) : \exists (\xi^0, \eta^0) : (\xi^0, \eta^0) \in C' \times K'[-g(x^0)], \\ (\xi^0, \eta^0) \neq (0, 0) \quad and \quad \langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle \ge 0 \,.$$

$$193$$

(Sufficient Conditions) Suppose that for each $u \in \ker h'(x^0) \setminus \{0\}$ the following condition is satisfied:

$$\begin{split} & (y^0, \, z^0) \in (f, \, g)'(x^0; u) : \exists (\xi^0, \, \eta^0) : (\xi^0, \, \eta^0) \in C' \times K'[-g(x^0)], \\ & (\xi^0, \, \eta^0) \neq (0, \, 0) \quad and \quad \langle \xi^0, \, y^0 \rangle + \langle \eta^0, \, z^0 \rangle > 0 \,. \end{split}$$

Then, x^0 is an *i*-minimizer of problem (8).

REFERENCES

[1] G. P. CRESPI, I. GINCHEV, M. ROCCA. First order optimality conditions in set-valued optimization, *Math. Methods Oper. Res.*, **63** (2006), 87–106.

[2] M. DUREA. First and second order optimality conditions for set-valued optimization problems, *Rend. Circ. Mat. Palermo*, **53** (2004), 451–468.

[3] I. GINCHEV, M. ROCCA. On constrained set-valued optimization, International Conference 60 years Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, July 6–8, 2007, Sofia, Bulgaria.

[4] I. GINCHEV, A. GUERRAGGIO, M. ROCCA. First-order conditions for $C^{0,1}$ constrained vector optimization, In: F. Giannessi, A. Maugeri (eds.), Variational analysis and applications, *Nonconvex Optim. Appl.*, **79**, New York, Springer, 2005, 427–450.

[5] J. JAHN, R. RAUH. Contingent epiderivatives and set-valued optimization, *Math. Methods Oper. Res.*, **46** (1997), 193–211.

[6] A. A. KHAN, F. RACITI. A multiplier rule in set-valued optimisation, *Bull. Austral. Math. Soc.*, **68** (2003), 93–100.

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ОПТИМИЗАЦИОННИ ЗАДАЧИ ЗА МНОГОЗНАЧНИ ФУНКЦИИ СЪДЪРЖАЩИ ЕДНОЗНАЧНИ ИЗОБРАЖЕНИЯ ОТ ТИП РАВЕНСТВА

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Разглежда се оптимизационна задача с многозначни функции $\min_{C} F(x), G(x) \cap (-K) \neq \emptyset, h(x) = 0$, където $C \subset \mathbb{R}^m$ и $K \subset \mathbb{R}^p$ са затворени изпъкнали конуси, $F : \mathbb{R}^n \to \mathbb{R}^m$ и $G : \mathbb{R}^n \to \mathbb{R}^p$ са многозначни функции и $h : \mathbb{R}^n \to \mathbb{R}^q$ е C^1 -функция. Третират се два типа решения, а именно *w*-минимуми (слабо ефективни решения) и *i*-минимуми (изолирани минимуми). С използването на многозначна производна по посока на Дини се извеждат необходими условия от първи ред една точка да бъде *w*-минимум и достатъчни условия от първи ред една точка да бъде *i*-минимум. Условията се формулират както в първична, така и в дуална форма.