## A FIRST-ORDER CHARACTERIZATION OF THE STRONGLY PSEUDOCONVEX FUNCTIONS*

Vsevolod Ivanov Ivanov<br>A characterization of the strongly pseudoconvex functions, introduced by Karamardian and Schaible [3], is derived. Relation between this class of functions and the differentiable strongly convex ones is considered.

1. Introduction. In this paper we consider the strongly pseudoconvex functions which were introduced by Karamardian and Schaible [3]. These functions have the property that the minimizer of a strongly pseudoconvex function over an open set is unique. Each strongly pseudoconvex function is strictly pseudoconvex, but it could be not convex.

We obtain a characterization of these functions. Then we show that the uniformly strongly pseudoconvex functions are exactly the strongly convex differentiable functions.

We denote the transpose of the vector $v$ by $v^{T}$, the gradient of the function $f$ at the point $x$ by $\nabla f(x)$, and the norm of the vector $v$ by $\|v\|$.
2. The main result. Recall the following notion introduced by Karamardian and Schaible [3].

Definition 1. A differentiable real function $f$ is called strongly pseudoconvex on an open set $S$ of the finite-dimensional Euclidean space $\mathbb{R}^{n}$ if there exists $\alpha>0$ such that, for all pairs of distinct points $x \in S, y \in S$, we have

$$
\begin{equation*}
(y-x)^{T} \nabla f(x) \geq 0 \quad \text { implies } \quad f(y) \geq f(x)+\alpha\|y-x\|^{2} . \tag{1}
\end{equation*}
$$

The following theorem provides a complete characterization of the strongly pseudoconvex functions.

Theorem 1. Let the differentiable real function $f$ be defined on the open set $S$ of $\mathbb{R}^{n}$. Then, $f$ is strongly pseudoconvex on $S$ if and only if there exists a non-negative function $p: S \times S \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
f(y) \geq f(x)+p(x, y)(y-x)^{T} \nabla f(x)+\alpha\|y-x\|^{2} \tag{2}
\end{equation*}
$$

for all $x \in S, y \in S$.

[^0]Proof. The sufficiency is obvious. We prove the necessity. Suppose that there exists $\alpha>0$ satisfying implication (1). We construct explicitly the function $p$ as follows:

$$
p(x, y)= \begin{cases}\frac{f(y)-f(x)-\alpha\|y-x\|^{2}}{(y-x)^{T} \nabla f(x)}, & \text { if } f(y)<f(x)+\alpha\|y-x\|^{2} \\ & \text { or }(y-x)^{T} \nabla f(x)>0 \\ 1, & \text { otherwise }\end{cases}
$$

The function $p$ is well-defined, non-negative, and it satisfies inequality (2). Indeed, according to the strong pseudoconvexity the sets

$$
\left\{(x, y) \in S \times S \mid(y-x)^{T} \nabla f(x)>0\right\}
$$

and

$$
\left\{(x, y) \in S \times S \mid f(y)<f(x)+\alpha\|y-x\|^{2}\right\}
$$

have an empty intersection. If $f(y)<f(x)+\alpha\|y-x\|^{2}$, then by the contrapositive form of implication (1) we have $(y-x)^{T} \nabla f(x)<0$ and $p$ is positive. If $(y-x)^{T} \nabla f(x)>0$, then $f(y) \geq f(x)+\alpha\|y-x\|^{2}$ because of the strong pseudoconvexity, and $p$ is non-negative. Otherwise,

$$
(y-x)^{T} \nabla f(x) \leq 0 \quad \text { and } \quad f(y) \geq f(x)+\alpha\|y-x\|^{2} .
$$

Therefore,

$$
\begin{gathered}
f(y)-f(x)-\alpha\|y-x\|^{2} \geq 0 \geq(y-x)^{T} \nabla f(x) \\
=p(x, y)(y-x)^{T} \nabla f(x)>0 .
\end{gathered}
$$

The proof is complete.
Consider the uniform case when the function $p$ is a non-negative constant $C$, i.e. it does not depend on $x, y$. Then,

$$
\begin{equation*}
f(y) \geq f(x)+C(y-x)^{T} \nabla f(x)+\alpha\|y-x\|^{2} \quad \text { for all } \quad x, y \in S \tag{3}
\end{equation*}
$$

The following notion was introduced by Polyak [5].
Definition 2. A real function $f$ is said to be strongly convex on an open convex set $S$ of $\mathbb{R}^{n}$ if there exists $\alpha>0$ such that

$$
f(\lambda y+(1-\lambda) x) \leq \lambda f(y)+(1-\lambda) f(x)-\alpha \lambda(1-\lambda)\|y-x\|^{2}
$$

for all $x \in S, y \in S$ and $\lambda \in[0,1]$.
It follows from (3) that

$$
\begin{equation*}
f(x) \geq f(y)+C(x-y)^{T} \nabla f(y)+\alpha\|x-y\|^{2} \quad \text { for all } \quad x, y \in S \tag{4}
\end{equation*}
$$

We obtain from (3), (4) that

$$
\begin{equation*}
(y-x)^{T}(\nabla f(y)-\nabla f(x)) \geq(2 \alpha / C)\|x-y\|^{2} \quad \text { for all } \quad x, y \in S \tag{5}
\end{equation*}
$$

The following theorem is well-known (see Polyak [5], Karamardian [2]).
Theorem 2. The differentiable function $f: S \rightarrow \mathbb{R}$ is strongly convex on the open convex set $S$ if and only if there exists $\kappa>0$ such that

$$
(y-x)^{T}(\nabla f(y)-\nabla f(x)) \geq \kappa\|x-y\|^{2} \quad \text { for all } \quad x, y \in S
$$

Therefore, the class of uniformly strongly pseudoconvex functions, that is the class
for which there exist $\alpha>0$ and $C>0$ satisfying (3), coincides with the differentiable strongly convex functions.

We compare our results for strongly pseudoconvex functions with similar ones for pseudoconvex functions.

Recall the following notion introduced by Mangasarian [4] in 1965: A differentiable real function $f$ is called pseudoconvex on an open set $S$ of the space $\mathbb{R}^{n}$ if

$$
x \in S, y \in S, f(y)<f(x) \quad \text { implies } \quad(y-x)^{T} \nabla f(x)<0 .
$$

The following theorem was recently derived by the author [1].
Theorem 3. The differentiable real function $f$ defined on the open set $S$ is pseudoconvex on $S$ if and only if there exists a positive function $p: S \times S \rightarrow \mathbb{R}$ such that

$$
f(y) \geq f(x)+p(x, y)(y-x)^{T} \nabla f(x) \quad \text { for all } \quad x, y \in S
$$

Using similar arguments, we obtain from the last inequality that the class of uniformly pseudoconvex functions, that is the class for which $p$ does not depend on $x$ and $y$, coincides with the differentiable convex functions.

## REFERENCES

[1] V. I. Ivanov. First order characterizations of pseudoconvex functions. Serdica Math. J., 27 (2001), 203-218.
[2] S. Karamardian. The nonlinear complementarity problem with applications, part 2. J. Optimization Theory Appl., 4 (1969), 167-181.
[3] S. Karamardian, S. Schaible. Seven kinds of monotone maps. J. Optimization Theory Appl., 66 (1990), 37-46.
[4] O. L. Mangasarian. Nonlinear programming. Repr. of the orig. 1969. Classics in Applied Mathematics, Volume 10. Philadelphia, PA: SIAM, 1994.
[5] B. T. Polyak. Existence theorems and convergence of minimization sequences in extremum problems with restrictions. Soviet Math., 7 (1966), 72-75.

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# ХАРАКТЕРИЗАЦИЯ ОТ ПЪРВИ РЕД НА СИЛНО ПСЕВДОИЗПЪКНАЛИТЕ ФУНКЦИИ 

## Всеволод Иванов Иванов

Получена е характеризация на силно псевдоизпъкналите функции, въведени от S. Karamardian и S. Schaible в [3]. Разгледана е връзката на този клас от функции с диференцируеми силно изпъкнали функции.


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