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COMPOUND POISSON COUNTING DISTRIBUTIONS*

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In this paper discrete compound Poisson distributions are given. The Inflated-parameter Poisson distribution, Poissin distribution of order k and Pólya-Aeppli distribution of order k are defined. The probability mass functions and recursion formulas are given. An interpretation of Pólya-Aeppli distibution of order k is considered. The Pólya-Aeppli process of order k is defined.

1. Introduction. The Poisson distribution belongs to the family of Generalized Power Series Distributions (GPSO) and is basically used for counting [6]. The probability generating function (PGF) is given by

$$P(t) = e^{\lambda(t-1)},$$

where $\lambda > 0$ is a parameter. In many cases it is of interest to obtain the probability distribution of a random sum of independent equally distributed random variables, for example, the claims payable by an insurance company.

The random variables considered are assumed to be defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Consider a random variable N that can be represented as

$$N = X_1 + X_2 + \ldots + X_Y, \tag{1}$$

where $Y, X_1, X_2, \ldots$ are mutually independent, non-negative, integer valued random variables. Then, the random variable N is said to have a *compound distribution*. The distribution of X is a compounding distribution. We suppose that the random variable Y is Poisson distributed. The PGF of the random variable N is given by

$$P_N(t) = e^{\lambda(P_X(t)-1)}, \tag{2}$$

where $P_X(t)$ is the PGF of the random variable X. The random variable N has a compound Poisson distribution and belongs to the family of Compound GPSDs.

In this note we consider three types of generalizations of the Poisson distribution by compounding. In Section 2 the Inflated-parameter Poisson distribution is given. In Section 3 we introduce the Poisson distribution of order k. In Section 4 the Pólya-Aeppli distribution of order k is defined. It is a compound Poisson distribution by truncated geometric compounding distribution. Finally, as an application, in Section 5 are given the properties of the Pólya-Aeppli process of order k.

2. Inflated-parameter Poisson distribution. In [7] and [8] the classical discrete distributions are generalized by including an additional parameter $\rho \in [0, 1)$. The new

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family of compound distributions is called Inflated – parameter generalized power series distributions (IGPSD). In this family the compounding distribution is geometric, $Ge(1-\rho)$, with parameter $\rho \in [0,1)$. The probability mass function (PMF) and the PGF are given by

$$P(X=i) = (1-\rho)\rho^{i-1}, \quad i = 1,2,\ldots, \tag{3}$$

and

$$P_X(t) = \frac{(1-\rho)t}{1-\rho t}. \tag{4}$$

In the case when $Y \sim Po(\lambda)$, we say that the r.v. N has an Inflated-parameter Poisson distribution with parameters $\lambda > 0$ and ρ ($N \sim IPo(\lambda,\rho)$). The PMF is given by

$$P(N=m) = \begin{cases} e^{-\lambda}, & m = 0, \\ e^{-\lambda}\sum_{i=1}^{m}\binom{m-1}{i-1}\frac{[(1-\rho)\lambda]^i}{i!}\rho^{m-i}, & m = 1,2,\ldots. \end{cases} \tag{5}$$

In the case when $\rho = 0$, the PMF (5) coincides with the PMF of the classical Poisson distribution.

The Inflated-parameter Poisson distribution coincides with the Pólya-Aeppli distribution, studied by Evans [4], see also [6].

The PMF of the Pólya-Aeppli distribution satisfies the following second order recurrent formula [8]:

$$p_m = \left(2\rho + \frac{\lambda(1-\rho)-2\rho}{m}\right)p_{m-1} - \left(1-\frac{2}{m}\right)\rho^2 p_{m-2},$$

for $m = 1,2,\ldots$ and $p_{-1} = 0$.

3. Poisson distribution of order k. The probability distributions of order k are introduced by A. Philippou, C. Georghiou and G. Philippou [11]. The geometric distribution of order k, ($Ge_k(p)$) is defined by the number of trials until the first occurrence of k consecutive successes in a sequence of independent trials with success probability p. The negative binomial ($NB_k(r,p)$) distribution of order k is the distribution of the sum of r independent, equally $Ge_k(p)$ distributed random variables. The Poisson distribution of order k, ($Po_k(\lambda)$), is a limiting distribution of a sequence of shifted NB_k distributed random variables.

Hirano [5] introduced the binomial distribution of order k ($Bi_k(n,p)$). Aki, Kuboku and Hirano [1] derived the logarithmic series distribution of order k, ($LS_k(p)$), as a limiting distribution of a sequence of NB_k distributed random variables. It is proved that the discrete distributions of order k can be represented as a Compound GPSDs, (see [2] and also [3]).

Let the random variable N have a compound distribution.

Definition 3.1. *If the compounding random variable X is a discrete distributed, truncated at 0 and from the right away from $k+1$, then the random variable N has a distribution of order k.*

The $Ge_k(p)$, $NB_k(r,p)$ and $LS_k(p)$ distributions belong to the family of Compound GPSDs by truncated geometric compounding distribution.

The random variable X has a PMF and PGF, given by

$$P(X=m)=\frac{1-\rho}{1-\rho^k}\rho^{m-1}, \quad m=1,2,\dots k, \tag{6}$$

and

$$P_X(t)=\frac{(1-\rho)t}{1-\rho^k}\frac{1-\rho^k t^k}{1-\rho t}, \tag{7}$$

where $\rho \in [0,1)$ and $k \geq 1$ is a fixed integer.

If $k \to \infty$, then the truncated geometric distribution asymptotically coincides with the $Ge(1-\rho)$ distribution, defined by (3) and (4).

In this way, the $Ge_k(p)$, $NB_k(r,p)$ and $LS_k(p)$ distributions converge to the corresponding IGPSD.

The $Po_k(\lambda)$ distribution is obtained by a discrete uniform compounding distribution.

4. The Pólya-Aeppli distribution of order *k*. In this section we introduce the Pólya-Aeppli distribution of order k, defined in [10]. It is a compound Poisson distribution with PGF given by (2). The compounding distribution is a truncated geometric with PMF and PGF, given by (6) and (7).

Definition 4.1. *The probability distribution defined by the PGF* (2) *and compounding distribution, given by* (6) *and* (7) *is called a Pólya-Aeppli distribution of order k, ($PA_k(\lambda,\rho)$).*

The Pólya-Aeppli distribution of order k belongs to the family of Compound GPSD, compounded by the truncated geometric distribution. The following theorem gives the probability mass function of the $PA_k(\lambda,\rho)$ distribution. Let us denote $Z=\dfrac{\lambda(1-\rho)}{1-\rho^k}$, for simplicity.

Theorem 4.1. *The probability mass function of the $PA_k(\lambda,\rho)$ distributed random variable is given by:*

$$p_0 = e^{-\lambda},$$

$$p_i = e^{-\lambda}\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{Z^j}{j!}\rho^{i-j}, \qquad i=1,2,\dots,k,$$

$$p_i = e^{-\lambda}[\textstyle\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{Z^j}{j!}\rho^{i-j} - Z\rho^k\sum_{j=0}^{i-k-1}\binom{i-k-1}{j}\frac{Z^j}{j!}\rho^{i-k-1-j}],$$

$$i=k+1,k+2,\dots,2k+1,$$

$$p_i = e^{-\lambda}[\textstyle\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{Z^j}{j!}\rho^{i-j} - Z\rho^k\sum_{j=0}^{i-k-1}\binom{i-k-1}{j}\frac{Z^j}{j!}\rho^{i-k-1-j}+$$

$$+\frac{(Z\rho^k)^2}{2!}\sum_{j=0}^{i-2k-2}\binom{i-2k-1}{j+1}\frac{Z^j}{j!}\rho^{i-2k-2-j}], \qquad i=2k+2,\dots,3k+2,$$

$$p_i = e^{-\lambda}[\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{Z^j}{j!}\rho^{i-j} - Z\rho^k\sum_{j=0}^{i-k-1}\binom{i-k-1}{j}\frac{Z^j}{j!}\rho^{i-k-1-j}$$
$$+\frac{(Z\rho^k)^2}{2!}\sum_{j=0}^{i-2k-2}\binom{i-2k-1}{j+1}\frac{Z^j}{j!}\rho^{i-2k-2-j} -$$
$$-\frac{(Z\rho^k)^3}{3!}\sum_{j=0}^{i-3k-3}\binom{i-3k-1}{j+2}\frac{Z^j}{j!}\rho^{i-3k-3-j}], \qquad i = 3k+3,\dots,4k+3,$$
$\dots$

Remark 4.1. If $k \to \infty$, then the Pólya-Aeppli distribution of order k asymptotically coincides with the usual Pólya-Aeppli distribution. If $\rho = 0$, then it is a Poisson distribution.

Remark 4.2. The mean and the variance of the Pólya-Aeppli distribution of order k are given by
$$EN = \frac{1+\rho+\ldots+\rho^{k-2}+\rho^{k-1}-k\rho^k}{1-\rho^k}\lambda$$
and
$$Var(N) = \frac{1+3\rho+5\rho^2+7\rho^3+\ldots+(2k-3)\rho^{k-2}+(2k-1)\rho^{k-1}-k^2\rho^k}{1-\rho^k}\lambda.$$

The following proposition gives an extension of the Panjer recurrence formulas.

Proposition 4.1. *The PMF of the Pólya-Aeppli distribution of order k satisfies the following recurrence formulas:*
$$p_1 = Zp_0,$$
$$p_i = \left(2\rho + \frac{Z-2\rho}{i}\right)p_{i-1} - \left(1-\frac{2}{i}\right)\rho^2 p_{i-2}, \; i = 2,3,\dots k,$$
$$p_i = \left(2\rho + \frac{Z-2\rho}{i}\right)p_{i-1} - \left(1-\frac{2}{i}\right)\rho^2 p_{i-2} - \frac{k+1}{i}Z\rho^k p_{i-k-1} + \frac{k}{i}Z\rho^{k+1}p_{i-k-2},$$
$$i = k+1, k+2, \dots,$$
where $p_{-1} = 0$.

Proof. Differentiation of (2) with $P_X(t)$, given by (7), leads to
$$(1-\rho t)^2 P_N'(t) = \frac{1-\rho}{1-\rho^k}[1-(k+1)\rho^k t^k + k\rho^{k+1}t^{k+1}]P_N(t),$$
where $P_N(t) = \sum_{m=0}^{\infty} p_m t^m$ and $P_N'(t) = \sum_{m=0}^{\infty}(m+1)p_{m+1}t^m$. The recurrence formulas are obtained by equating the coefficients of t^m on both sides for fixed $m = 0,1,2,\dots$. $\square$

5. Pólya-Aeppli process of order *k*. Let $N(t)$ represents the state of the system at time $t \geq 0$. It is assumed that the process has state space $\mathcal{N}$, the non-negative integers. Let $\lambda > 0$ be any real number and $\rho \in [0,1)$.

Suppose that $N(t)$ has a $PA_k(\lambda t, \rho)$ distribution. Then, the PGF of $N(t)$ is given by
$$h(u,t) = e^{\lambda t[P_X(u)-1]}, \tag{8}$$

where $P_X(u)$ is the PDF of the truncated geometric distribution. The properties of the defined process are specified by the following postulates:

$$P(N(t+h)=n \mid N(t)=m)=\begin{cases} 1-\lambda h+o(h), & n=m, \\ \dfrac{1-\rho}{1-\rho^k}\rho^{i-1}\lambda h+o(h), & n=m+i, i=1,2,\dots,k \end{cases}$$

for every $m=0,1,\dots$, where $o(h)\to 0$ as $h\to 0$. Note that the postulates imply that for $i=k+1,k+2,\dots,\ P(N(t+h)=m+i \mid N(t)=m)=o(h)$.

Let $P_m(t)=P(N(t)=m),\ m=0,1,2,\dots$. Then, the above postulates yield the following Kolmogorov forward equations:

$$\begin{aligned} &P_0'(t)=-\lambda P_0(t), \\ (9)\qquad &P_m'(t)=-\lambda P_m(t)+\frac{1-\rho}{1-\rho^k}\lambda\sum_{j=1}^{m\wedge k}\rho^{j-1}P_{m-j}(t), \quad m=1,2,\dots, \end{aligned}$$

with the conditions

$$P_0(0)=1 \quad \text{and} \quad P_m(0)=0, \quad m=1,2,\dots. \tag{10}$$

Multiplying the m-th equation of (9) by u^m and summing for all $m=0,1,2,\dots$, we get the following differential equation

$$\frac{\partial h(u,t)}{\partial t}=-\lambda[1-P_X(u)]h(u,t). \tag{11}$$

The solution of (11) with the initial condition

$$h(u,0)=1$$

is given by (8), which is the PGF of the $PA_k(\lambda t,\rho)$ distribution.

Definition 5.1. *The process defined by* (9) *and* (10) *is called a Pólya-Aeppli process of order k.*

Remark 5.1. *In the case of $k\to\infty$, the Pólya-Aeppli process of order k approaches to the Pólya-Aeppli proces, defined by Minkova* [9]. *If $\rho=0$, then it is a homogeneous Poisson process.*

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СЛОЖНИ ПОАСОНОВИ БРОЯЩИ РАЗПРЕДЕЛЕНИЯ

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В работата се разглеждат дискретни сложни Поасонови разпределения. Дефинирани са Поасоново разпределение с инфлационен параметър, Поасоново разпределение от ред k и разпределение на Пойа-Аепли от ред k. Дадени са вероятностните функции и рекурентни формули за вероятностите. Накрая е дадена интерпретация на разпределението на Пойа-Аепли от ред k. Дефиниран е и процес на Пойа-Аепли от ред k.